Inverse Problems of Tomographic Type

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Applied Inverse Problems, Pre-conference workshop
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Inverse Problems of Tomographic Type
Direct and inverse problems

Direct problem: given input $f$ and operator $A$, find the output
$$g = Af$$

$$f \xrightarrow{A} g := Af$$

Inverse problem: given (a variety of) input(s) $f$ and output(s) $g = Af$, find the operator $A$

$$f \xrightarrow{A = ?} g := Af$$
Questions to ask

- Injectivity of the direct transform input \( f \) ⇒ output \( g \). (Uniqueness of reconstruction)
- Inversion algorithms
- Stability of inversion
- Range of the direct operator
- Incomplete data effects
- Contrast
- Resolution
Tomography

- Derived from the Greek tomos (part) and graphein (to write).
- Has existed since 1950s, and is still alive and kicking (more than before)
- Applied in medicine (diagnostics and treatment), biology, geophysics, archeology, astronomy, material science, industry, oceanography, atmospheric sciences, homeland security ...
- Inexhaustible source of wonderful hard mathematical problems. Involves Differential Equations, Numerical Analysis, Integral, Differential, and Algebraic Geometry, Several Complex Variables, Probability/Statistics, Number Theory, Discrete Mathematics... IS JUST FUN!
Examples of tomographic reconstructions

In medicine
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Examples of tomographic reconstructions

Industry

Inverse problems
X-ray tomography = CAT scan
Radon transform
Invariances of the Radon transform

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Geophysics
CAT scan

CAT = Computer Assisted Tomography
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A. Cormack and G. Hounsfield built the first computed
tomography scanners in 1960s, won 1979 Nobel prize in medicine.
CAT scan (＝X ray tomography) procedure

The X-ray tomography (＝CAT scan) procedure:

\[ l_0, l_1 - \text{initial and terminal intensities.} \]
Beer’s law

\[ I(x) \] – intensity of X-ray beam traveling along line \( L \) at point \( x \).

Beer’s law: \( dl = -f(x)I(x)dx \),

where \( f(x) \) – attenuation coefficient at \( x \)

\[
\frac{dl(x)}{dx} = -f(x)I(x) \Rightarrow \frac{d\ln I(x)}{dx} = -f(x) \Rightarrow I_1 = e^{-\int_L f(x)dx} I_0
\]

I.e., measurements provide \( \int_L f(x)dx \) for any line \( L \)

The density plot of \( f(x) \) is the **tomogram**.
Transport equation

\[ u(x, \omega) \text{ density of particles at } x \text{ moving in the direction } \omega. \]

\[ \omega \cdot \nabla_x u(x, \omega) + f(x)u(x, \omega) = \int \sigma(s, \omega', \omega)u(x, \omega')d\omega' + s(x). \]

\( f(x) \) - attenuation coefficient, \( \sigma \) -scattering coefficient, \( s(x) \) -sources density.

In absence of scattering and sources, one gets Beer’s law.
Given a function $f(x)$ Radon transform produces the values of $\int_L f(x)dx$ along all lines $L$:

$$Rf(L) := \int_L f(x)dx$$

In coordinates:
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Inverse problems
X-ray tomography = CAT scan
Radon transform
Invariances of the Radon transform

Radon transform in coordinates

$$Rf(\omega, s) := \int_{x \cdot \omega = s} f(x) dx, (\omega, s) \in C := S^1 \times \mathbb{R}$$
Symmetries

- **Evenness** \( Rf(\omega, s) = Rf(-\omega, -s) \)
  since \( x \cdot \omega = s \iff x \cdot (-\omega) = -s \)
  Identify \((\omega, s) \sim (-\omega, -s)\)
  \(g := Rf\) is defined on the M"obius strip \((S \times \mathbb{R})/ \sim\)

- **Shift invariance.**
  Let \((T_a f)(x) := f(x + a), (t_p g)(\omega, s) := g(\omega, s + p)\)
  \(R(T_a f)(\omega, s) = t_a \cdot \omega Rf = Rf(\omega, s + a \cdot \omega)\)

**Exercise:** Prove this.

**Corollary:** Fourier transform methods might be useful
Symmetries continued

- **Rotation invariance.**
  
  \[ R(f(Ax))(\omega, s) = Rf(A\omega, s) \]

  **Exercise:** Prove this.

  **Corollary:** Fourier series expansions might be useful.

- **Dilation invariance** (check it) \( \Rightarrow \) Melin transform is useful.
Fourier transforms

- \( f(x) \) on \( \mathbb{R}^2 \)
  \[
  \tilde{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x) e^{-ix \cdot \xi} \, dx
  \]
  \[
  f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(\xi) e^{ix \cdot \xi} \, d\xi
  \]

- \( g(s) \) on \( \mathbb{R} \)
  \[
  \hat{g}(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s) e^{-i s \sigma} \, ds
  \]
  \[
  g(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{g}(\sigma) e^{i s \sigma} \, d\sigma
  \]
Ridge functions

Ridge function $Q(x) = q(x \cdot \omega)$

Coupling with ridge functions

$$\int_{\mathbb{R}^2} f(x) Q(x) dx = \int_{\mathbb{R}} Rf(\omega, s) q(s) ds.$$ 

For $Q(x) = e^{i\xi \cdot x} = e^{i\sigma \omega \cdot x}$ get

$$\int_{\mathbb{R}^2} f(x) e^{-i\sigma \omega \cdot x} dx = \int_{\mathbb{R}} Rf(\omega, s) e^{-i\sigma s} ds$$

Projection-slice (Fourier-slice) formula

$$\tilde{f}(\sigma \omega) = \frac{1}{\sqrt{2\pi}} \hat{g}(\omega, \sigma)$$
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Uniqueness

Any compactly supported function $f$ is uniquely determined by $Rf$. Indeed, $Rf(\omega, s)$ determines the Fourier transform of $f$, and thus $f$ itself.
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Dual Radon

$R : f(x) \mapsto Rf(\omega, s) - \text{Radon},$

$R^\# : g(\omega, s) \mapsto R^\# g(x) - \text{dual Radon transform}$, such that

$$\int Rf(\omega, s)g(\omega, s)d\omega ds = \int f(x)R^\# g(x)dx$$
Exercise: Prove the formula:

\[ R^\# g(x) = \int_S g(\omega, x \cdot \omega) d\omega \]

The lines with normal coordinates \((\omega, x \cdot \omega)\) are passing through \(x\):

This explains the name backprojection operator.
**d-plane transform**

In $\mathbb{R}^n$ ($n > 2$), one can take $d$-dimensional Radon transform for any $1 \leq d < n$:

$$f(x) \mapsto R_d f(H) = \int_{H} f(x)dx,$$  
where $H$ – $d$-dimensional subspace

E.g., in $\mathbb{R}^3$:

- $d = 2$ - Radon transform $\int_{x \cdot \omega = s} f(x)dx$
- $d = 1$ - X-ray transform $\int_{-\infty}^{\infty} f(x + t\omega)dt$
Projection-slice formula gives an inversion procedure:

\[ f(x) \Rightarrow Rf(\omega, s) \Rightarrow \hat{Rf}(\omega, \sigma) = \sqrt{2\pi} \hat{f}(\sigma\omega) \Rightarrow f(x) \]
Let’s elaborate on Fourier inversion, denoting $g := Rf$:

\[
f(x) = \frac{1}{2\pi} \int e^{i\xi \cdot x} \tilde{f}(\xi) d\xi = \frac{1}{2\pi} \int_{|\omega|=1} \int_0^\infty e^{i\sigma\omega \cdot x} \tilde{f}(\sigma\omega) |\sigma| d\sigma d\omega
\]

\[
= \frac{1}{4\pi} \int_{|\omega|=1}^{\infty} \ldots \cdot \frac{1}{4\pi} \int_{|\omega|=1}^{\infty} d\omega \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma\omega \cdot \hat{g}(\omega, \sigma)|\sigma|} d\sigma \right)
\]

\[
= \frac{1}{4\pi} R^\# \left( H \frac{dg}{ds} g \right)(x) = \frac{1}{4\pi} R^\# H \frac{dg}{ds} g(x).
\]
Filtered backprojection (FBP) inversion

FBP inversion

\[ f = \frac{1}{4\pi} R^\# H \frac{d}{ds} Rf, \]  where \( H \frac{d}{ds} \) - filtration, \( R^\# \) - backprojection

Exercise

Compute \( R^\# Rf \) (i.e., filtration omitted) and see why it produces a blurred version of \( f \).
An example of a Matlab FBP inversion

Phantom (left) and its reconstruction from 128 projections and 128 detectors. USE SIMPLE, BUT REVEALING PHANTOMS!
Polar coordinates $r, \omega$ (i.e., $x = r\omega$) on the plane, 
$\omega = (\cos \phi, \sin \phi)$. Standard coordinates $(\omega, s)$ in Radon domain. 
If $f(x)$ - the original function, $g(\omega, s) = Rf$ -its Radon image, expand into Fourier series w.r.t. $\phi$:

$$f(r, \omega) = \sum_{l} f_l(r) e^{il\phi}$$
$$g(\omega, s) = \sum_{l} g_l(r) e^{il\phi}$$

Any good formulas for $\{f_l\} \Leftrightarrow \{g_l\}$?
Fourier series continued - Cormack’s formulas

- Rotational invariance ⇒ $g_l$ depends on $f_l$ only.
- **Exercise**: Straightforward calculation:

$$g_l(s) = 2 \int_0^\infty \left(1 - \frac{s^2}{r^2}\right)^{-1/2} T_{|l|} \left(\frac{s}{r}\right) f_l(r) dr,$$

where $T_l(x) = \cos(l \arccos(x))$ - Chebyshev polynomial of 1st kind.

$$f_l(r) = -\frac{1}{\pi} \int_0^\infty (s^2 - r^2)^{-1/2} T_{|l|} \left(\frac{s}{r}\right) g'_l(s) ds.$$
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Range

$Q$: is a given $g(\omega, s)$ Radon transform of some $f$? (appropriate function classes assumed)

$A$: Let $g = Rf$.

i) $g(\omega, s) = g(-\omega, -s)$

ii) consider $k$th **moment** $G_k(\omega) := \int_{\mathbb{R}} s^k g(\omega, s) ds$ - function on the unit sphere.

$$G_k(\omega) = \int_{-\infty}^{\infty} ds \int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}^2} (x \cdot \omega)^k f(x) dx$$

homogeneous polynomial of degree $k$ in $\omega$
Range conditions

- **Evenness.** $g(\omega, s) = g(-\omega, -s)$

- **Moment conditions.** For any $k = 0, 1, 2, \ldots$, $G_k(\omega)$ extends to a homogeneous polynomial of degree $k$ of $\omega \in \mathbb{R}^2$. 
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Fourier transform inversion
Filtered backprojection inversion
Fourier series inversion (Cormack’s formulas)

**Range conditions**

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**Range revisited in Fourier domain**

Consider 1D FT \(\hat{g}(\omega, \sigma)\) of \(g = Rf\). Get 2D FT of \(f\) along radial lines:

\[
\tilde{f}(\sigma \omega) = c \hat{g}(\omega, \sigma)
\]

Notice the role of evenness!

Function \(h(\sigma \omega) := \hat{g}(\omega, \sigma)\) is smooth everywhere, possibly except the origin.

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**Exercise**

- Prove that smoothness of \(h\) at zero implies that \(\frac{d^k \hat{g}}{ds^k}(\omega, 0)\) extends to a homogeneous polynomial of degree \(k\) in \(\omega\).
- Prove that this is equivalent to the moment conditions on \(g\).
A problem of finding $g$ from $f$ is correctly posed (in Hadamard’s sense), if it has unique solution and small variations in $f$ lead to small variations in $g$. Otherwise the problem is ill-posed. This will be specified in various settings throughout the workshop. The characteristic property of inverse problems is their usual ill-posedness.
Notion of stability

Stability:
Small variations in data lead to small changes in the result.

SVD reduces any linear problem to solving $Ax = y$ with a large diagonal matrix $A$:

$$
\begin{pmatrix}
ad_1 & 0 & 0 & \ldots & 0 \\
0 & a_2 & 0 & \ldots & 0 \\
0 & 0 & a_3 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\ldots \\
\end{pmatrix}
=
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\ldots \\
\end{pmatrix}
$$

When $a_k \to 0$ fast - instability.
Stability of Radon inversion

### Stability estimate for Radon transform in 2D

| $D$ - disk in $\mathbb{R}^2$, $f \in H^s_0(D)$, $C_1 \| Rf \|_{H^{s+1/2}} \leq \| f \|_{H^s} \leq C_2 \| Rf \|_{H^{s+1/2}}$ |

### Conclusion:

X-ray transform smoothes functions by $1/2$ derivatives. Same with the dual. Thus, two of them add a derivative, which must be removed during inversion (remember the filter $H \frac{d}{ds}$?).

More generally, $d$-plane transform adds $d/2$ derivatives, thus in FBP $d$ derivatives need to be removed by filtration.
Incomplete data

What if the values of $Rf(\omega, s)$ are known for a set of points $(\omega, s)$ only?

- Is the reconstruction possible (uniqueness)?
- Reconstruction procedures.
- Stability.
- Any resulting image deterioration?
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Limited angle problem
Exterior problem
Interior problem
Singularity detection

Limited angle

Data known for an open set of $\omega s$ and all $s$.

Any compactly supported function is uniquely reconstructed.

Follows from Paley-Wiener theorem.

Corollary:

Data known for an angle of $\omega s$ and all $s \Rightarrow$ compactly supported functions are uniquely reconstructed.

Reconstruction is unstable.
A limited angle reconstruction

Limited angle problem
Exterior problem
Interior problem
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ROI (Region Of Interest) imaging - can we use only beams intersecting the ROI?

**Exterior problem**: data is known for beams not intersecting a (bounded convex) domain (i.e., ROI is the exterior):

- Reconstruction is unique, if the function decays at $\infty$ faster than any power $|x|^{-N}$ (not true for a fixed power, no matter how large). Reconstruction is unstable.
Example of exterior reconstruction

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Exterior problem
Interior problem
Singularity detection
Interior problem

ROI is the interior of a domain:

Reconstruction is non-unique. \textbf{But} ....
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Example of interior reconstruction

Phantom

Interior Data Reconstruction
Visible/audible singularities

Q.: Why did some of the interfaces disappear?
A.: Since they are “invisible” with the limited data we had.

Rule of thumb:

Wave front point \((x, \xi)\) is “invisible” if there is no line (surface) of integration in the data set passing through \(x\) and co-normal to \(\xi\).

Q.: What about uniqueness theorems? Everything should be visible?
A.: “Visible” theoretically, but unstable (and thus impossible) to reconstruct.
Local tomography

Making the filtration in FBP stronger, emphasizes (but does not move, create, or eliminate) the singularities of $f$. E.g., in 2D consider the following modified reconstruction operator:

$$Ag := R^\# \frac{d^2 g}{ds^2}$$

(notice the replacement of $H \frac{d}{ds}$ by $\frac{d^2}{ds^2}$).

**Theorem**

$AR$ is an elliptic operator of order 1. Thus $WF(ARf) = WF(f)$, but singularities of $f$ are emphasized in $ARf$. 
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Local reconstruction example - limited angle
Local reconstruction example - exterior data
Local reconstruction example - interior data
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Other transforms of Radon type arising in tomography  

- $S$ - surface in $\mathbb{R}^3$ or curve in $\mathbb{R}^2$. Spherical mean operator  

$$f(x) \mapsto M_S f(p, r) := \int_{|\theta|=1} f(p + r\theta) \, d\theta, \quad p \in S, \, r \in \mathbb{R}_+$$

arises in thermo- and photo-acoustic tomography.  

- In $\mathbb{R}^3$ - integrals over cones with vertices on a surface $S$. Arises in Compton camera imaging.  

- Integrals over horocycles and geodesics in the hyperbolic plane - arise in electrical impedance imaging.  

- Generalizations to Riemannian manifolds, to tensor fields rather than functions, etc.
A classification of tomographic procedures

- Transmission (X-ray, Ultrasound, Optical, Electrical Impedance)
- Emission (SPECT, PAT, astronomy, homeland security)
- Reflection (Ultrasound)
- What’s that ????

Emission tomography and attenuated X-ray transform

MRI
SPECT = Single Photon Emission Computed Tomography

Imaging self-radiating objects (medicine, nuclear reactors, security).

\( f(x) \) - unknown source distribution, \( \mu(x) \) - attenuation.

**Attenuated Radon transform**

\[
f \mapsto R_\mu f(\omega, s) = \int f(x) e^{-\int_{L_x} \mu(y) dy} \, dx
\]
Emission tomography - continued

- Notice evenness disappear: $g(\omega, s) \neq g(-\omega, -s)$
- After 30 years long history, uniqueness, inversion, range, and stability issues have finally been resolved very recently.
- **Q.**: Can one recover simultaneously $f$ and $\mu$?
  **A.**: ??? (Recent major advances by Bal and Bukhgeim)
Physics and mathematics of MRI are too complex to discuss now. The bottom line: 2D or 3D Fourier transform of a function (tomogram) is measured. Then the tomogram is reconstructed either by direct FT inversion, or by reducing (projection-slice f-la) to the Radon transform and then inverting it.
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\[
\begin{align*}
- \nabla \cdot \sigma(x) \nabla u &= 0 \text{ in } \Omega \\
u \mid_{\partial \Omega} &= f, \sigma \frac{\partial u}{\partial \nu} \mid_{\partial \Omega} &= g \\
\Lambda : f &\to g - \text{Dirichlet-to-Neumann map}
\end{align*}
\]
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**EIT - Pros and Contras**

Great contrast 😊
Low resolution (exponential instability) 😞
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- $\nabla \cdot \sigma(x) \nabla u + \mu u = 0 = 0$ in $\Omega$

Great contrast 😊 Low resolution (exponential instability) 😞
No justice in the world!

There seem to be no cheap, safe, high contrast, and high resolution methods! 😞

Q: What can one do?
A: Plagiarise! (Tom Lehrer) Sorry! Combine, hybridize.
Now the **hybrid methods** come into play!
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Thermo-/photo-acoustic imaging and spherical mean operators
Helping EIT: Acousto Electric Tomography, MREIT, CDII
Helping OT: Ultrasound Modulated Optical Tomography
Magnetic resonance elastography (MRE)

TAT/PAT - thermo/photo-acoustic tomography

Agranovsky, Ambartsoumian, Ammari, Arridge, Bal, Burgholzer,
Cox, Finch, Haltmeier, Hristova, Kang, Kuchment, Kunyansky,
Nguyen, Palamodov, Patch, Quinto, Rakesh, Scherzer, Stefanov,
Uhlmann, Wang, Xu, ...

Recovering \( f(x) \) from its restricted spherical means \( M_S f(p, r) \).
Standard issues resolved recently: uniqueness, inversion, stability,
The truth about TAT

If \(c(x)\) - sound speed, the model is:

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = c^2(x)\Delta u, & \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \\
u(0, x) = f(x), \frac{\partial u}{\partial t}(0, x) = 0
\end{cases}
\]

Inversion of the operator \(f \mapsto g := u \big|_{s \times \mathbb{R}_+}\)

Reduces to spherical means for constant sound speed only.
Related to spectral theory, transmission eigenvalues, spectral geometry, number theory
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Thermo-/ photo-acoustic imaging and spherical mean operators
Helping EIT: Acousto Electric Tomography, MREIT, CDII
Helping OT: Ultrasound Modulated Optical Tomography
Magnetic resonance elastography (MRE)

More about TAT

Reconstruction methods available:

- **FBP formulas** - available in all dimensions, stable; work only for constant speed, known for $S$ - sphere only, do not work when $f$ extends beyond $S$.

- **Eigenfunction expansions** - all dimensions, stable, (theoretically) for variable speed, $S$ - arbitrary, work when $f$ can extend beyond $S$; probably unfeasible numerically for variable speed.

- **Time reversal** - all dimensions, stable, easy to implement, variable speed, any $S$, any location of $f$. 
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Incomplete data and singularities
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Other types of tomography involving integral geometric transforms
A classification of tomographic procedures
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Some TAT reconstructions

Thermo-/photo-acoustic imaging and spherical mean operators
Helping EIT: Acousto Electric Tomography, MREIT, CDII
Helping OT: Ultrasound Modulated Optical Tomography
Magnetic resonance elastography (MRE)
Acousto Electric Tomography (AET)

Ammari, Bal, Bonnetier, Capdebosque, Fink, Kang, Kuchment, Kunyansky, Scherzer, Steinhauer, Wang, Xu, ...

Send acoustic waves!

Keep currents steady!

Measure potentials
CAT scan and Radon/X-ray transform
Relations with the Fourier transform.
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AET - reconstructions

Ammari, Bal, Bonnetier, Capdebosq, Fink, Kang, Kuchment, Kunyansky, Scherzer, Steinhauer, Wang, Xu, ...

Phantom
Noiseless reconstruction
50% noise reconstruction
CAT scan and Radon/X-ray transform
Relations with the Fourier transform. Dual Radon
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UOT

Allmaras, Bal, Bangerth, Dobson, Kuchment, Nam, Oraevsky,
Schotland, Steinhauer, Uhlmann, Wang, ...
A combination of OT with ultrasound irradiation, similarly to AET.

Peter Kuchment
Inverse Problems of Tomographic Type
Magnetic resonance elastography (MRE)

Ehman, Manduca, J. McLaughlin, ...
Using MRI data to recover mechanical properties of biological tissues (e.g., stiffness).
Seo et al (MREIT), Joy, Nachman, Tamasan, Timonov ...

(CDI=Current density imaging)

MRI data are used in conjunction with EIT to arrive to a stable mathematical problem and good reconstructions.
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