Geometric methods for inverse problems for elliptic equations

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in Inverse Problems Research

1 Inverse problem in applications

Calderón's inverse problem: Measure electric resistance between all boundary points of a body. Can the conductivity be determined in the body?

Inverse problem for the wave equation: Let us send waves from the boundary of a body and measure the waves at the boundary. Can the wave speed be determined in the body?

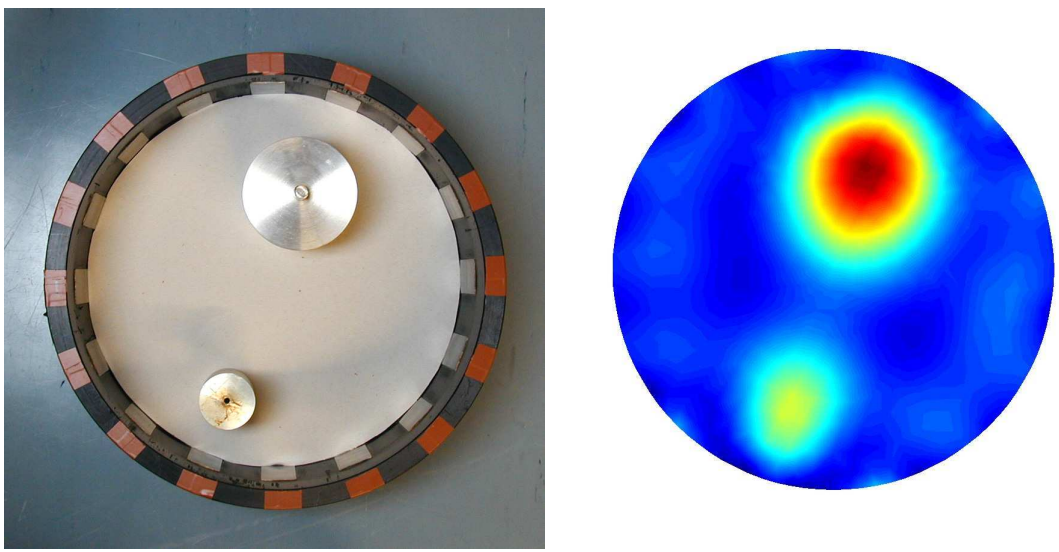


Figure: University of Kuopio.

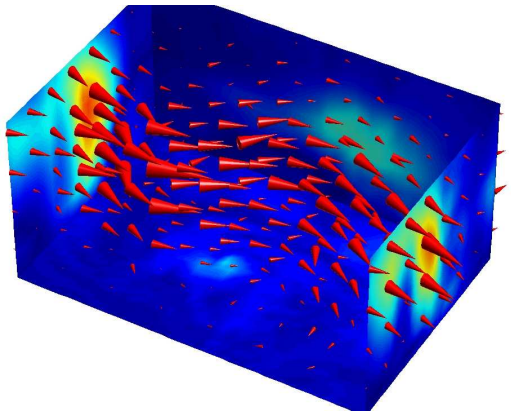
2 Inverse conductivity problem

Consider a body $\Omega \subset \mathbb{R}^n$. An electric potential $u(x)$ causes the current

$$J(x) = -\sigma(x)\nabla u(x).$$

Here the conductivity $\sigma(x)$ can be an isotropic, that is, scalar, or an anisotropic, that is, matrix valued function. If the current has no sources inside the body, we have

$$\nabla \cdot \sigma(x)\nabla u(x) = 0.$$



Conductivity equation

$$\nabla \cdot \sigma(x) \nabla u(x) = 0 \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = f.$$

Calderón's inverse problem: Do the measurements made on the boundary determine the conductivity, that is, does $\partial\Omega$ and the Dirichlet-to-Neumann map Λ_σ,

$$\Lambda_\sigma(f) = \nu \cdot \sigma \nabla u|_{\partial\Omega}$$

determine the conductivity $\sigma(x)$ in Ω?

Let u^f be the solution of

$$\begin{aligned} &\nabla\cdot\sigma(x)\nabla u(x) = 0 \quad \text{in } \Omega, \\ &u|_{\partial\Omega} = f. \end{aligned}$$

The Dirichlet-to-Neumann map $\Lambda_\sigma f = \nu\cdot\sigma\nabla u|_{\partial\Omega}$ defines the quadratic form

$$Q_\sigma[f,h] = \int_{\partial\Omega} f\Lambda_\sigma h\, dS.$$

We have

$$\begin{aligned} Q_\sigma[f,h] &= \int_\Omega \sigma\nabla u^f\cdot\nabla u^h\, dx, \\ Q_\sigma[f,f] &= \inf_{w|_{\partial\Omega}=f}\int_\Omega \sigma\nabla w\cdot\nabla w\, dx. \end{aligned}$$

Some previous results for inverse conductivity problem:

- Calderón 1980: Solution of the linearized inverse problem.
- Sylvester-Uhlmann 1987: Uniqueness of inverse problem in $\mathbb{R}^n$, $n \geq 3$
- Nachman 1996: Calderón's problem in $\mathbb{R}^2$
- Astala-Päivärinta 2003: Uniqueness of Calderón's problem in $\mathbb{R}^2$ with L^∞-conductivity
- Sylvester 1990: Inverse problem for an anisotropic conductivity near constant in $\mathbb{R}^2$.
- Siltanen-Mueller-Isaacson 2000: Explicit numerical solution for the 2D-inverse problem.
- Kenig-Sjöstrand-Uhlmann 2006: Reconstructions with limited data.

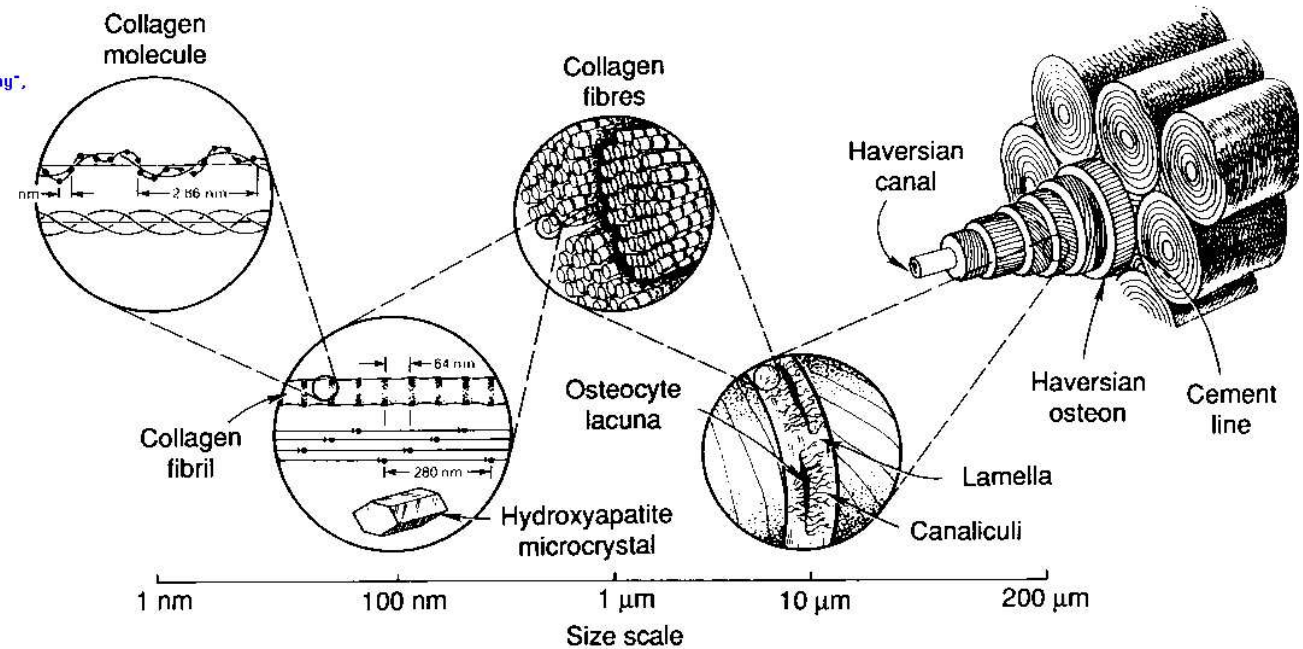
3 Anisotropic inverse problems

- Non-uniqueness.
- Invariant formulation. Uniqueness and non-uniqueness results
- Applications to Euclidean space: non-uniqueness results.

Anisotropy appears in homogenization of isotropic material

Lakes, R. S., "Materials with structural hierarchy", Nature, 361, 511–515 (1993).

FIG. 4 Hierarchical structure in human compact bone; individual size scales adapted from refs 28–31. Fibrous, laminar, particulate and porous structure is present at different size scales.

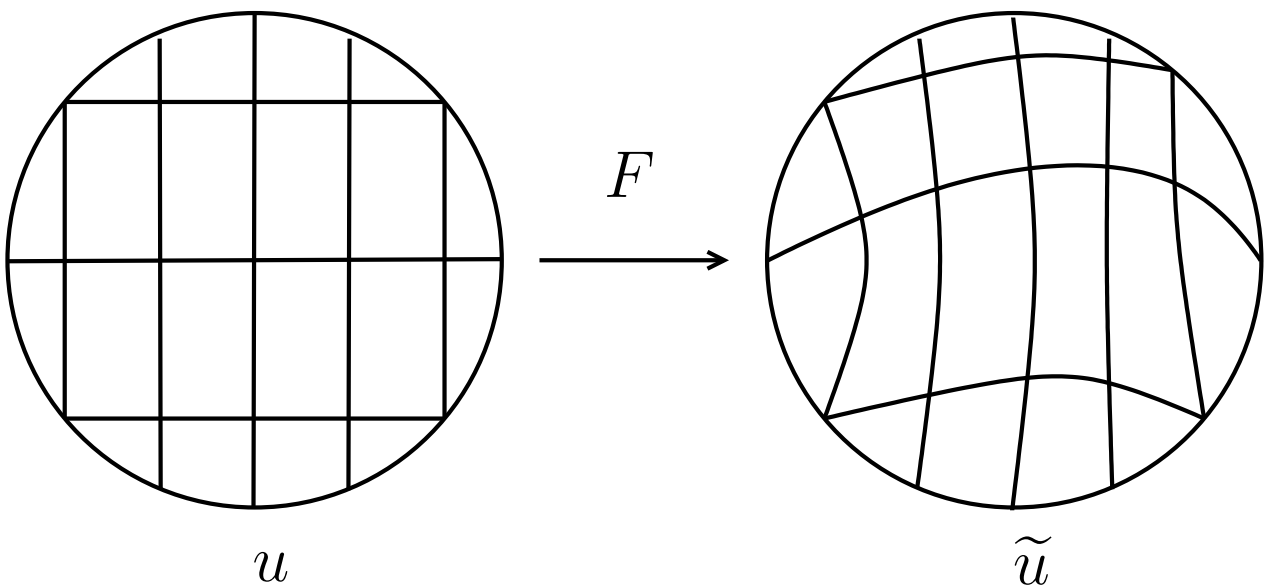


Structure of human bone.

Deformation of the domain. Assume that $\sigma(x) = (\sigma^{jk}(x)) \in \mathbb{R}^{n\times n}$. Let F be diffeomorphism

$$F : \Omega \to \Omega, \quad F|_{\partial\Omega} = Id.$$

Consider $\widetilde{u} \in C^1(\Omega)$ and $u(x) = \widetilde{u}(F(x))$.



Deformation of the domain. Assume that $\sigma(x) = (\sigma^{jk}(x)) \in \mathbb{R}^{n\times n}$. Let F be diffeomorphism

$$F : \Omega \to \Omega, \quad F|_{\partial\Omega} = Id.$$

Consider $\widetilde{u} \in C^1(\Omega)$ and $u(x) = \widetilde{u}(F(x))$.
Chain rule gives $Du(x) = D\widetilde{u}(F(x))\, DF(x)$, or

$$\nabla u(x) = DF(x)^t\, (\nabla\widetilde{u})(F(x)).$$

Next we use this to study the quadratic form Q_σ corresponding to the Dirichlet-to-Neumann map Λ_σ. Recall that

$$Q_\sigma[f,h] = \int_{\partial\Omega} f\Lambda_\sigma h\, dS,$$

$$Q_\sigma[f,f] = \inf_{u|_{\partial\Omega}=f} \int_\Omega \sigma\nabla u\cdot\nabla u\, dx.$$

Thus when $y = F(x)$ and $\nabla u(x) = DF^t(x) \cdot (\nabla \widetilde{u})(y)$,

$$\begin{aligned} & \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla u(x)\, dx \\ = & \int_\Omega \sigma(x) DF(x)^t\, (\nabla \widetilde{u})(F(x)) \cdot DF(x)^t\, (\nabla \widetilde{u})(F(x))\, dx \\ = & \int_\Omega \left[DF(x) \sigma(x) DF(x)^t \right]_{x=F^{-1}(y)} \nabla \widetilde{u}(y) \cdot \nabla \widetilde{u}(y)\, J dy \end{aligned}$$

where $J = \det DF^{-1}(y) = \dfrac{1}{\det (DF)(F^{-1}(y))}$ satisfies $dx = J dy$.

We define

$$\widetilde{\sigma}(y) = F_* \sigma(y) = \left. \frac{(DF) \cdot \sigma \cdot (DF)^t}{\det (DF)} \right|_{x=F^{-1}(y)} .$$

Denoting $y = F(x)$ and $u(x) = \widetilde{u}(F(x))$ and

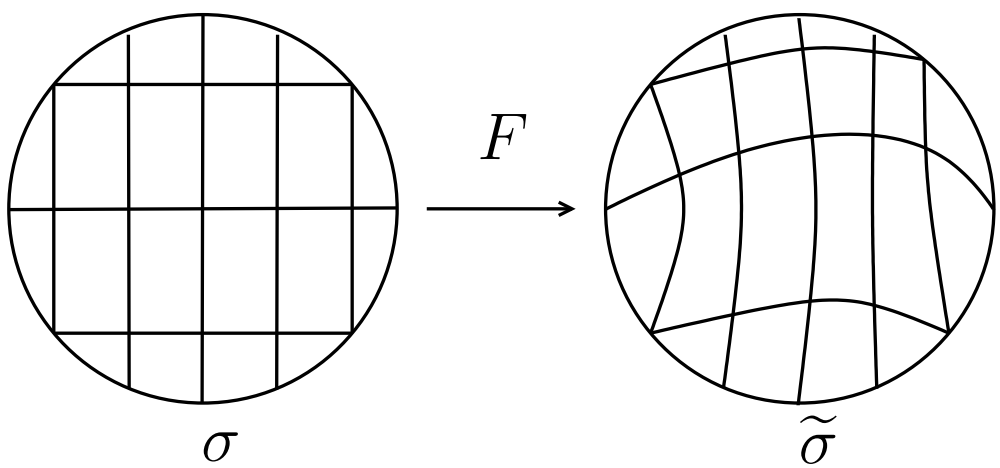
$$\widetilde{\sigma}(y) = F_*\sigma(y) = \left.\frac{(DF)\cdot\sigma\cdot(DF)^t}{\det(DF)}\right|_{x=F^{-1}(y)}$$

we have

$$Q_\sigma[f,f] = \inf_{u|_{\partial\Omega}=f}\int_\Omega \sigma(x)\nabla u(x)\cdot\nabla u(x)\,dx$$

$$= \inf_{\widetilde{u}|_{\partial\Omega}=f}\int_\Omega \widetilde{\sigma}(y)\nabla\widetilde{u}(y)\cdot\nabla\widetilde{u}(y)\,dy = Q_{\widetilde{\sigma}}[f,f].$$

Then $\Lambda_{\widetilde{\sigma}} = \Lambda_\sigma$.



Deformation of the domain and solutions. Assume that $\sigma(x) = (\sigma^{jk}(x)) \in \mathbb{R}^{n\times n}$,

$$\nabla\cdot\sigma\nabla u = 0 \quad \text{in } \Omega.$$

Let F be diffeomorphism

$$F : \Omega \to \Omega, \quad F|_{\partial\Omega} = Id.$$

Then

$$\nabla\cdot\widetilde{\sigma}\nabla\widetilde{u} = 0 \quad \text{in } \Omega,$$

where

$$\widetilde{u}(x) = u(F^{-1}(x)), \quad \widetilde{\sigma}(y) = F_*\sigma(y) = \left.\frac{(DF)\cdot\sigma\cdot(DF)^t}{\det(DF)}\right|_{x=F^{-1}(y)}$$

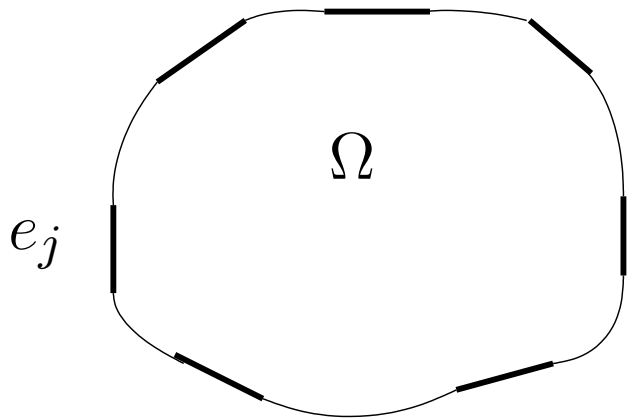
This observation is the basis of the transformation optics.

Electrical Impedance Tomography with an unknown boundary

Practical task: In medical imaging one often wants to find an image of the conductivity, when the domain Ω is poorly known.

Figure: Rensselaer Polytechnic Institute.

Complete electrode model. Let $e_j \subset \partial\Omega$, $j = 1, \ldots, J$ be disjoint open sets (electrodes) and



$$\begin{aligned} &\nabla\cdot \sigma\nabla v = 0 \quad \text{in } \Omega, \\ &z_j \nu\cdot \sigma\nabla v + v|_{e_j} = V_j, \\ &\nu\cdot \sigma\nabla v|_{\partial\Omega\setminus\cup_{j=1}^J e_j} = 0. \end{aligned}$$

Here z_j are the contact impedance of electrodes and $V_j \in \mathbb{R}$. The boundary measurements are the currents

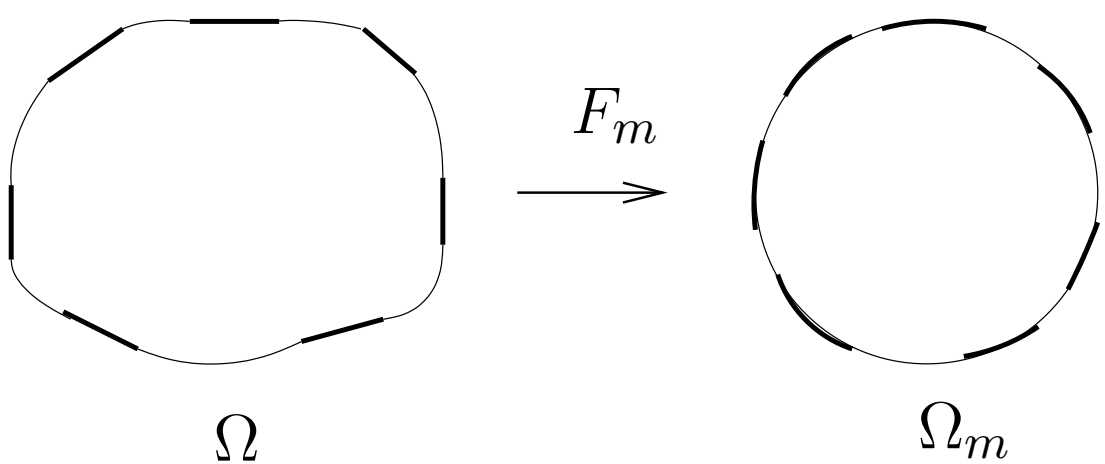
$$I_j = \frac{1}{|e_j|}\int_{e_j} \nu\cdot \sigma\nabla v(x)\, ds(x), \quad j = 1, \ldots, J.$$

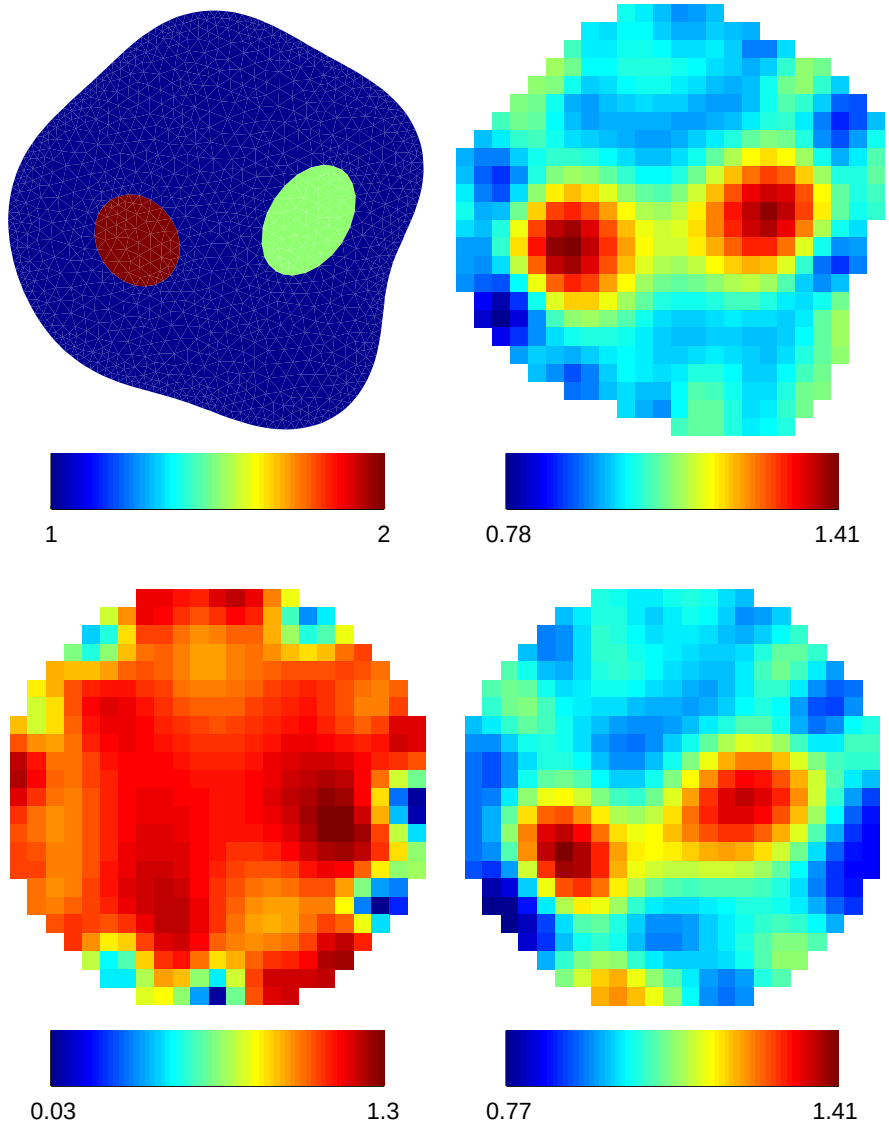
The matrix $E : (V_j)_{j=1}^J \to (I_j)_{j=1}^J$ is the electrode measurement matrix.

Mathematical formulation of EIT with unknown boundary:

1. Assume that σ is an isotropic conductivity in Ω.
2. Assume that we are given a set Ω_m that is our best guess for Ω. Let $F_m : \Omega \to \Omega_m$ be a map corresponding to the modeling error.
3. The given data is the electrode measurement matrix $E \in \mathbb{R}^{J \times J}$.

Fact: The deformation $F_m : \Omega \to \Omega_m$ can change an isotropic conductivity to an anisotropic conductivity.





Kolehmainen-L.-Ola 2005: Reconstruction of conductivity with the minimal anisotropy.

Algorithm:
By Teichmüller theory for quasiconformal mappings, for any σ there exists $\hat{\sigma}$ such that $\Lambda_{\widehat{\sigma}} = \Lambda_\sigma$ and there is $c_0 \in \mathbb{R}_+$ such that the eigenvalues of the matrix $\widehat{\sigma}(x)$ satisfy

$$\frac{\lambda_1(x)}{\lambda_2(x)} = c_0, \quad \text{for all } x \in \Omega_m. \tag{1}$$

Moreover, $\widehat{\sigma} = \widehat{F}_*(\sigma)$.
Using this we approximated the solution of the inverse problem using the regularized minimization problem

$$\min_{c_0, \widehat{\sigma}} \|E_{\widehat{\sigma}} - E_{measured}\|^2 + \alpha_1 |c_0 - 1|^2 + \alpha_2 \|\widehat{\sigma} - I\|^2_{H^1}.$$

where $\widehat{\sigma}$ satisfies (1).

Invariant formulation.
Assume $n \geq 3$. Consider Ω as a Riemannian manifold with

$$g^{jk}(x) = (\det \sigma(x))^{-1/(n-2)} \sigma^{jk}(x).$$

Then conductivity equation is the Laplace-Beltrami equation

$$\Delta_g u = 0 \quad \text{in } \Omega, \quad \text{where}$$

$$\Delta_g u = \sum_{j,k=1}^{n} g^{-1/2} \frac{\partial}{\partial x^j} (g^{1/2} g^{jk} \frac{\partial}{\partial x^k} u)$$

and $g = \det(g_{ij})$, $[g_{ij}] = [g^{jk}]^{-1}$.
Inverse problem: Can we determine the Riemannian manifold (M, g) by knowing ∂M and

$$\Lambda_{M,g} : u|_{\partial M} \mapsto \partial_\nu u|_{\partial M}.$$

Generally, solutions of anisotropic inverse problems are not unique. However, if we have enough a priori knowledge of the form of the conductivity, we can sometimes solve the inverse problem uniquely.

$$\begin{tikzcd}
 & \text{Manifold } (M, g) \arrow[dr] & \\
\text{Boundary measurements} \arrow[ur] & & \sigma^{jk}(x) \text{ on } \Omega \subset \mathbb{R}^n
\end{tikzcd}$$

Uniqueness results

Theorem 1 (L.-Taylor-Uhlmann 2003) *Assume that* (M, g) *is a complete* n*-dimensional real-analytic Riemannian manifold and* $n \geq 3$*. Then* ∂M *and*

$$\Lambda_{M,g} : u|_{\partial M} \mapsto \partial_\nu u|_{\partial M}$$

determine (M, g) *uniquely.*

Theorem 2 (L.-Uhlmann 2001) *Assume that* (M, g) *is a compact* 2*-dimensional Riemannian manifold. Then* ∂M *and*

$$\Lambda_{M,g} : u|_{\partial M} \mapsto \partial_\nu u|_{\partial M}$$

determine conformal class

$$\{(M, \alpha g) : \quad \alpha \in C^\infty(M),\ \alpha(x) > 0\}$$

uniquely.

We consider next the proof in the dimension $n = 2$.

Conformal manifolds

Assume that (M, g) and $(\widetilde{M}, \widetilde{g})$ conformal 2-dimensional manifolds, that is,

$$\widetilde{g}_{ij}(x) = \alpha(x) g_{ij}(x),$$

where $\alpha|_{\partial M} = 1,\ \alpha > 0$. Recall that

$$\Delta_g u = \sum_{j,k=1}^{n} g^{-1/2} \frac{\partial}{\partial x^j} (g^{1/2} g^{jk} \frac{\partial}{\partial x^k} u)$$

As $n = 2$ we have

$$\Delta_{\alpha g} u = \alpha^{-1} \Delta_g u$$

and the harmonic functions for g and $\widetilde{g}$ are the same. Thus

$$\Lambda_{g,\Gamma} = \Lambda_{\widetilde{g},\Gamma}.$$

Hence the best one can reconstruct is the conformal class.

Embedding M into a bigger manifold. We consider the 2-dimensional case $n = 2$ with $\Gamma = \partial M$.

Theorem 3 *(Lee-Uhlmann 1989) the pair* $(\partial M, \Lambda_g)$ *determines derivatives of the metric tensor in boundary normal coordinates in the case* $n \geq 3$ *and* $g_{\partial M}$ *in the case* $n = 2$*.*

We continue (M, g) to the manifold $(\widetilde{M}, g)$ where

$$\widetilde{M} = M \cup (\partial M \times [0, 1])$$

and

$$g|_{\partial M \times [0,1]} = g_{\partial M} \otimes ds^2$$

Green's functions

Let $G(x,y)$ be the Green's functions of $\widetilde{M}$,

$$\Delta_g G(\cdot\,,y) = \delta_y(\cdot\,) \text{ in } \widetilde{M},$$
$$G(\cdot\,,y)|_{\partial\widetilde{M}} = 0.$$

We denote

$$h_y(x) = G(x,y).$$

When $\widetilde{M} \setminus M$ and Λ_g are given, we can find $h_y(x)$, for $x, y \in \widetilde{M} \setminus M$ by solving the equation

$$\Delta_g h_y = \delta_y \text{ in } \widetilde{M} \setminus M,$$
$$h_y|_{\partial\widetilde{M}} = 0,$$
$$\partial_\nu h_y|_{\partial M}) = \Lambda_g(h_y|_{\partial M}).$$

Isothermal coordinates

Let $z \in \widetilde{M}^{int}$ and its small neighborhood U so that there is a harmonic function $\phi_1 : U \to \mathbb{R}$,

$$\Delta_g \phi_1 = 0.$$

If U is small enough, ϕ_1 has a harmonic conjugate, $\phi_2 : U \to \mathbb{R}$, satisfying

$$d\phi_2 = *d\phi_1, \quad \text{or} \quad \frac{\partial}{\partial x^j}\phi_2 = \sum_{i,k=1}^{2} det(g)^{1/2} g^{ik} \epsilon_{kj} \frac{\partial}{\partial x^i}\phi_1$$

Then ϕ_2 is harmonic and $\phi(x) = (\phi_1(x), \phi_2(x))$ defines in U coordinates and in these coordinates, the metric tensor is

$$g_{jk}(x) = \rho(x)\delta_{jk}, \quad \rho(x) \in \mathbb{R}_+.$$

Lemma 3.1 *Every* $z \in \widetilde{M}$ *has coordinate neighborhood* (U, ϕ), $\phi : U \to \mathbb{R}^2$ *such that*

$$\phi_* g = \rho(x) I, \ \rho(x) \in \mathbb{R}_+, \ I = (\delta_{ij}).$$

Moreover, in $\phi(U)$ *the functions*

$$h_y \circ \phi^{-1} : \phi(U) \to \mathbb{R}, \quad y \notin U$$

are real analytic.

Proof: Let ϕ be the standard isothermal coordinates. Then

$$\Delta_I (h_y \circ \phi^{-1}) = \rho \Delta_{\rho I} (h_y \circ \phi^{-1}) = 0.$$

Hence $h_y \circ \phi^{-1}$ are harmonic in $\mathbb{R}^2$-sense. $\square$

Lemma 3.2 *If the Jacobian of the map* $H = H_{y_1,y_2}$*,*

$$H_{y_1,y_2} : x \mapsto (h_{y_1}(x), h_{y_2}(x)),\ x \in A \subset \widetilde{M}$$

is non-zero, then functions

$$h_y \circ H^{-1} : H(A) \subset \mathbb{R}^2 \to \mathbb{R}$$

are real analytic.

Unique continuation principle and symmetry $G(x,y) = G(y,x)$ gives

Lemma 3.3 *If* $x, \widetilde{x} \in \widetilde{M}$ *are such that* $G(x,\,\cdot\,) = G(\widetilde{x},\,\cdot\,)$ *in an open set, then* $x = \widetilde{x}$*.*

Green's functions determine coordinates

Let $V \subset \widetilde{M} \backslash M$ be a small open set. For $x \in \widetilde{M}$, consider the infinite sequence

$$\Psi(x) = (h_y(x))_{y \in V}, \quad h_y(x) = G(x, y).$$

For $x_0 \in \widetilde{M}$ there is a neighborhood U and $y_1, y_2 \in V$ so that

$$x \mapsto (h_{y_1}(x), h_{y_2}(x)), \quad x \in U$$

are coordinates in U. Next we will consider function Ψ in these coordinates, that is,

$$(h_{y_1}(x), h_{y_2}(x)) \mapsto \Psi(x) = (h_y(x))_{y \in V}.$$

The components of this function are real analytic. We can cover $\widetilde{M}$ using these kind of neighborhoods.

Construction of the collection of local coordinate charts

Idea of construction

1. Let $V \subset \widetilde{M} \backslash M$ be a neighborhood of $\partial \widetilde{M}$ and $W_0 \subset \widetilde{M} \backslash (M \cup V)$. Find

$$h_y|_{W_0}, \; y \in V.$$

2. Choose $y_1, y_2 \in V$ such that $H = (h_{y_1}, h_{y_2})$ define regular coordinates $H : W_0 \to \mathbb{R}^2$.
3. Continue the real analytic functions

$$h_y \circ H^{-1}, \quad y \in V$$

from the set $H(W_0)$ to $\mathbb{R}^2$ as long as H defines coordinates and $h_y > 0$. Let $U_1 \subset \mathbb{R}^2$ be a largest set where the continuation can be done. Let $W_1 = H^{-1}(U_1)$.

4. Iterate the step 3: Choose new $y_1^{(m)}, y_2^{(m)} \in V$, $m = 1, 2, \ldots$ and denote $H_m = H_{y_1^{(m)}, y_2^{(m)}}$.
In the previous step we have found

$$h_y \circ H_{m-1}^{-1} : U_{m-1} \to \mathbb{R}, \quad y \in V.$$

Thus we can find in some open set the functions

$$h_y \circ H_m^{-1} = h_y \circ H_{m-1}^{-1} \circ (H_{m-1} \circ H_m^{-1}).$$

By finding a maximal real analytic continuation we obtain sets $U_m \subset \mathbb{R}^2$ and functions

$$h_y \circ H_m^{-1} : U_m \to \mathbb{R}, \quad y \in V$$

5. After the previous steps, the sets $W_m = H_m^{-1}(U_m)$ cover $\widetilde{M}$ and (W_m, H_m) are local coordinates on $\widetilde{M}$.

6. Substituting functions h_y, for $y \in V$ in the equations

$$\Delta_g h_y = \sum_{j,k=1}^{2} g^{-1/2} \frac{\partial}{\partial x^j} (g^{1/2} g^{jk} \frac{\partial}{\partial x^k} h_y) = 0$$

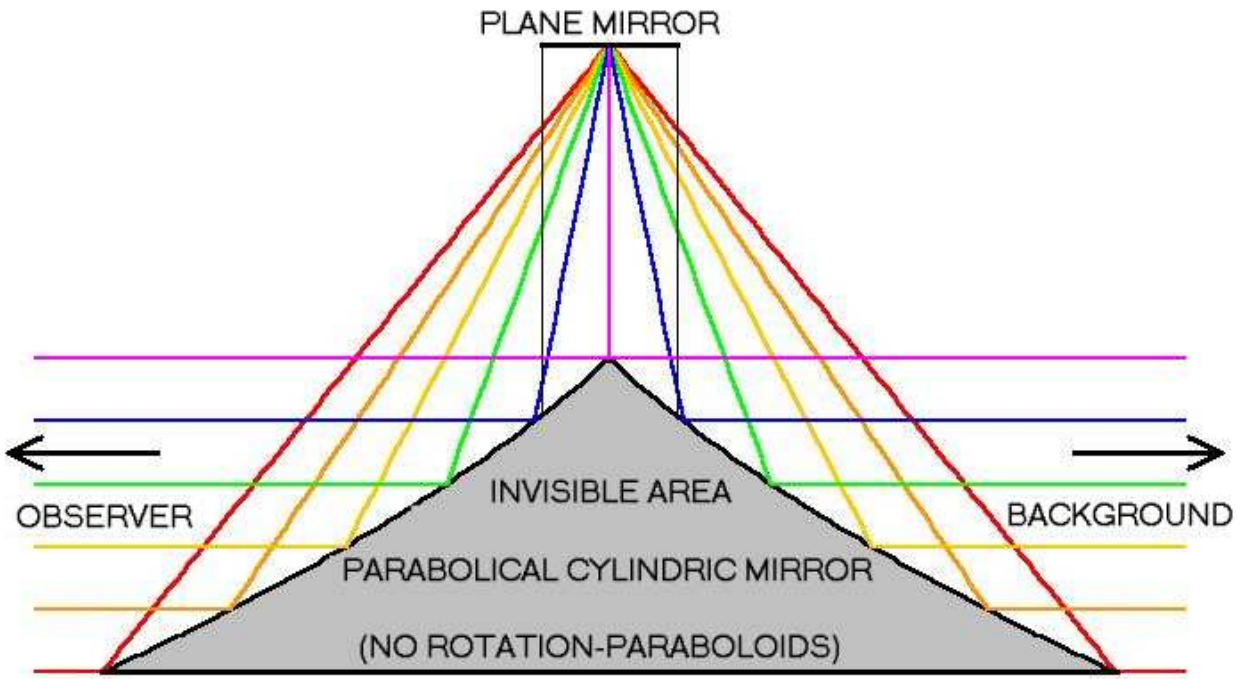
   we can find the representation of the metric tensor $g_{jk}(x)$ in the H_m coordinates up to a conformal factor $\alpha(x)$.

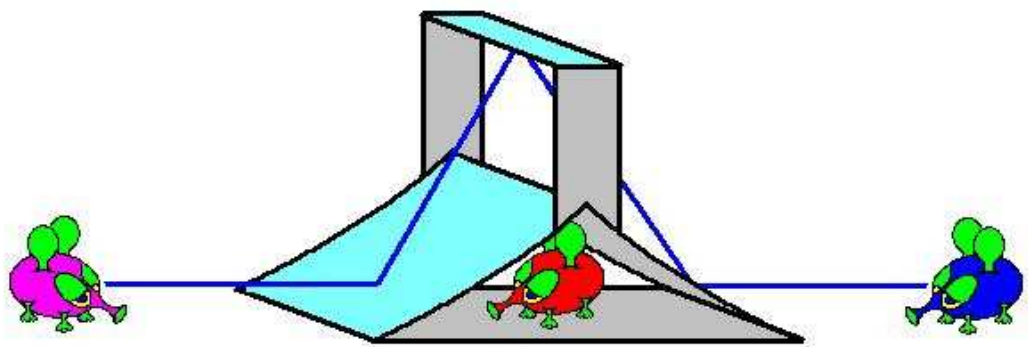
7. We identify points $x \in U_m$ and $x' \in U_{m'}$ if

$$h_y \circ H_m^{-1}(x) = h_y \circ H_{m'}^{-1}(x') \quad \text{for all } y \in V.$$

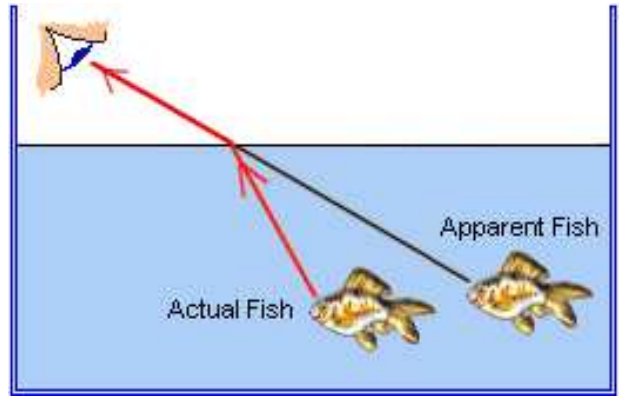
   In this way, the local coordinates can be glued together.

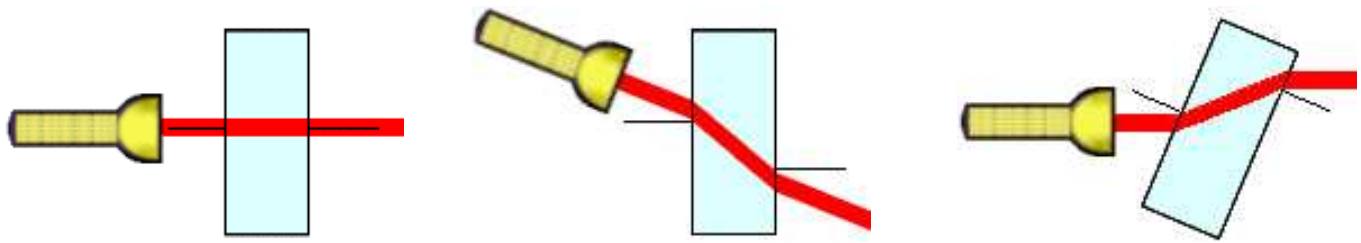
Invisibility cloaking with mirrors





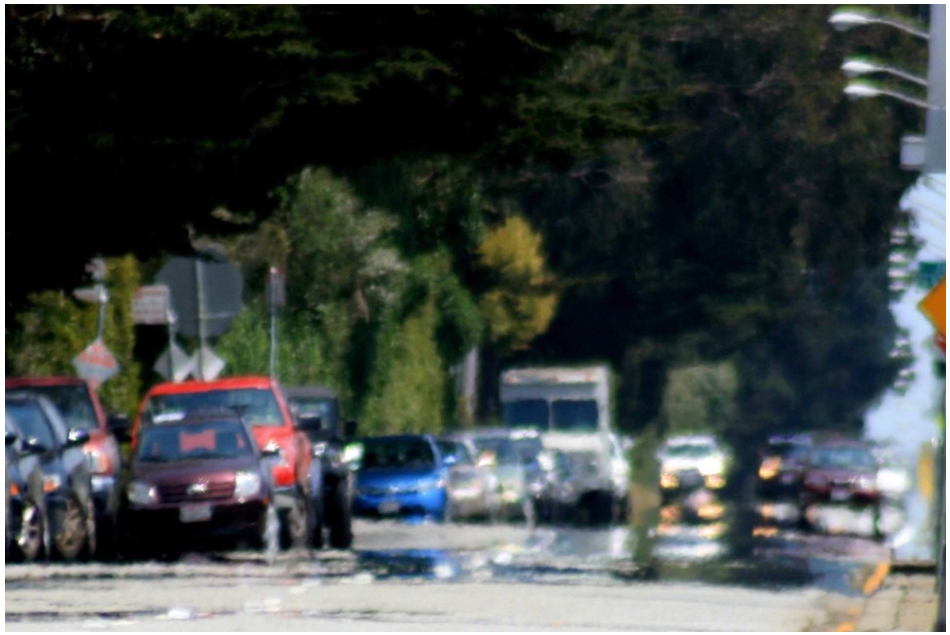
Illusions produced by changes of the speed of light.

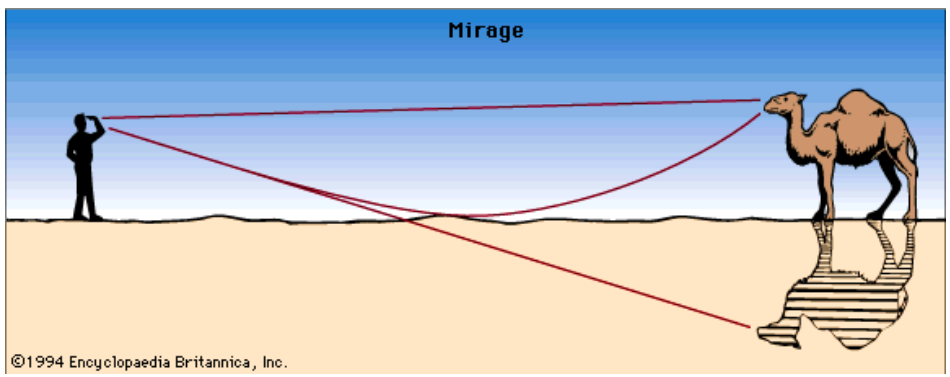




Figures: Williamson labs.

Mirage





Figures: Mila Zinkova and Encyclopaedia Britannica.

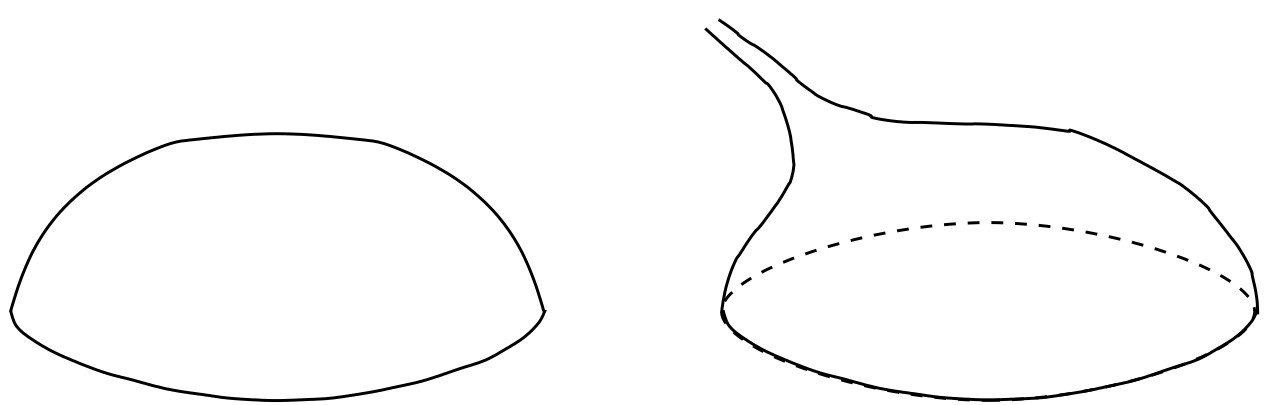
Non-uniqueness results (L.-Taylor-Uhlmann 2003)
Let (M, g) be a compact 2-dimensional manifold. Let $x_0 \in M$, and consider manifold

$$\widetilde{M} = M \setminus \{x_0\}$$

with the metric

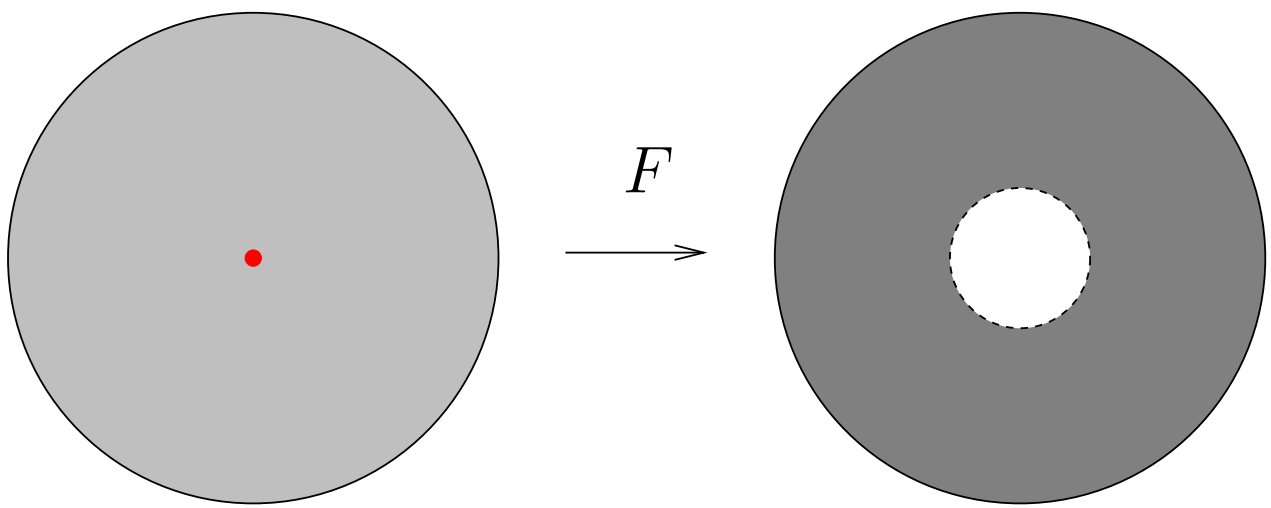
$$\widetilde{g}_{ij}(x) = \frac{1}{d_M(x, x_0)^2} g_{ij}(x).$$

Using Brownian motion, one can see that $\Lambda_{M,g} = \Lambda_{\widetilde{M},\widetilde{g}}$.



A ball with a cloaked pocket.
(Greenleaf-L.-Uhlmann 2003)
Let $B(2) \subset \mathbb{R}^3$ be a ball of radius 2 and $B(1)$ a ball of radius 1. Consider the map

$$F : B(2) \setminus \{0\} \to B(2) \setminus \overline{B}(1), \quad F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}.$$



We denote

$$\widetilde{g} = F_* g, \quad \text{that is,} \quad \widetilde{g}_{jk}(y) = \sum_{p,q=1}^{n} \frac{\partial x^p}{\partial y^j} \frac{\partial x^q}{\partial y^k} g_{pq}(x).$$

Let $g_{jk} = \delta_{jk}$ be the Euclidian metric in $B(2)$ and $\sigma = 1$ be the corresponding conductivity. Using $\widetilde{g} = F_* g$ we define

$$\widetilde{\sigma} = \begin{cases} |\widetilde{g}|^{1/2}\widetilde{g}^{jk} & \text{for } x \in B(2) \setminus \overline{B}(1) \\ \delta^{jk} & \text{for } x \in B(1). \end{cases}$$

Then in the spherical coordinates we have

$$\widetilde{\sigma} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}, \quad 1 < |x| \leq 2.$$

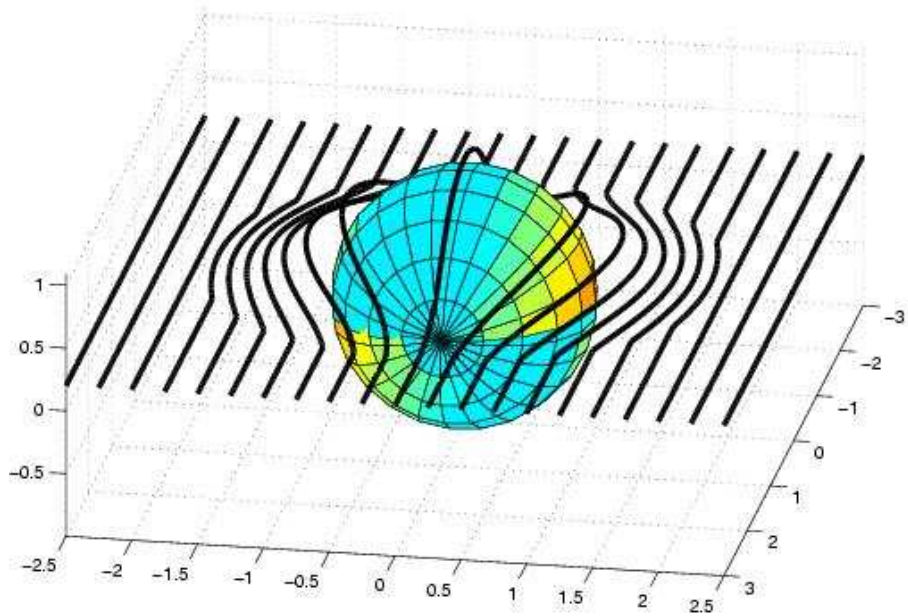
On boundary of $N = B(2)$ we define the Cauchy data set of H^1-solutions,

$$C_1(\widetilde{\sigma}) = \{(u|_{\partial N}, \nu \cdot \widetilde{\sigma}\nabla u|_{\partial N}) \ : \ u \in H^1(N), \ \nabla \cdot \widetilde{\sigma}\nabla u = 0\}.$$

Recall that $\gamma = 1$ in $B(1) \subset \mathbb{R}^3$ and

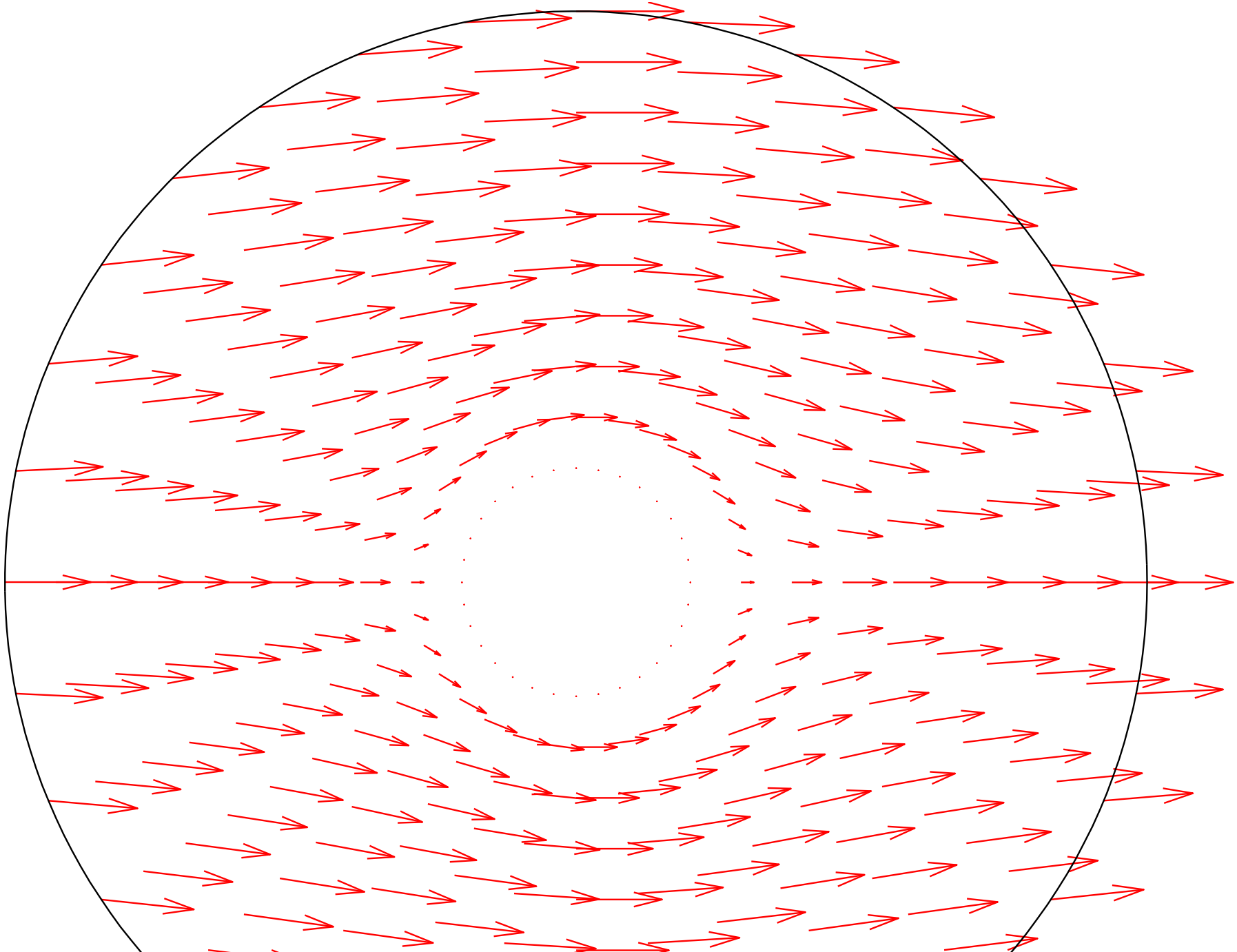
$$C_1(\widetilde{\sigma}) = \{(u|_{\partial N}, \nu \cdot \widetilde{\sigma} \nabla u|_{\partial N}) \ : \ u \in H^1(N), \ \nabla \cdot \widetilde{\sigma} \nabla u = 0\}.$$

Theorem 4 *(Greenleaf-L.-Uhlmann 2003) The boundary measurements for $\widetilde{\sigma}$ and γ coincide, that is, $C_1(\widetilde{\sigma}) = C_1(\gamma)$.*



Cloaking for the 2-dimensional conductivity equation is analyzed in Kohn-Shen-Vogelius-Weinstein 2007.

Currents go around the body



The electromagnetic community used the same singular metric for Maxwell equations that was introduced in Greenleaf-L.-Uhlmann 2003.

- Controlling electromagnetic fields by Pendry, Schurig and Smith, Science 2006.
- Metamaterial Electromagnetic Cloak at Microwave Frequencies by Schurig et al, Science 2006.

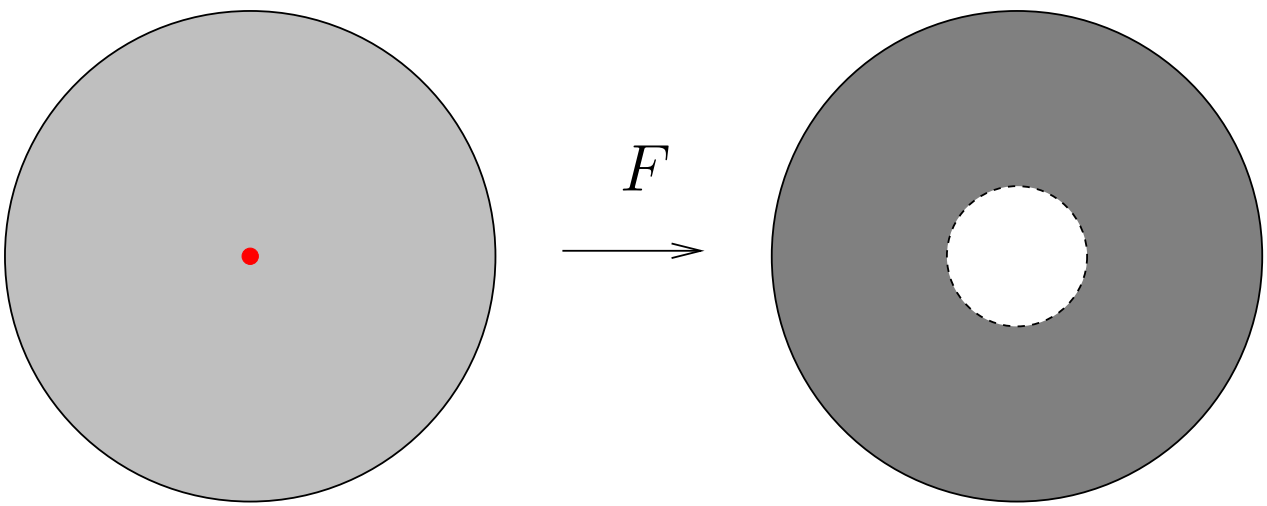
Other proposals for cloaking:

- Optical Conformal Mapping by Leonhardt, Science 2006.
- On the cloaking effects associated with anomalous localized resonance by Milton and Nicorovici, Proc. Royal Soc. 2006.
- Transparency with plasmonic and metamaterial coatings by Alu and Engheta, Phys. Rev. E 2005

The principle of cloaking for Maxwell's equations

As before, let $g_{ij} = \delta_{ij}$ and $\widetilde{g} = F_* g$ with

$$F : B(2) \setminus \{0\} \to B(2) \setminus \overline{B}(1), \quad F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}.$$



Define the permittivity $\widetilde{\epsilon}$ and permeability $\widetilde{\mu}$ by

$$\widetilde{\epsilon} = \widetilde{\mu} = \begin{cases} |\widetilde{g}|^{1/2}\widetilde{g}^{jk} & \text{for } x \in B(2) \setminus \overline{B}(1) \\ \delta^{jk} & \text{for } x \in B(1) \end{cases}$$

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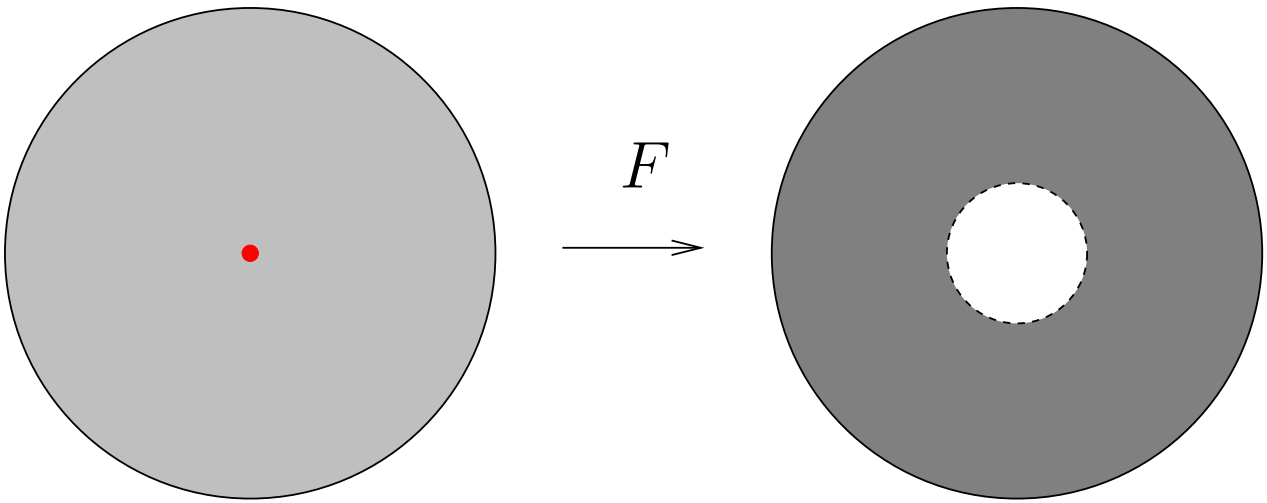

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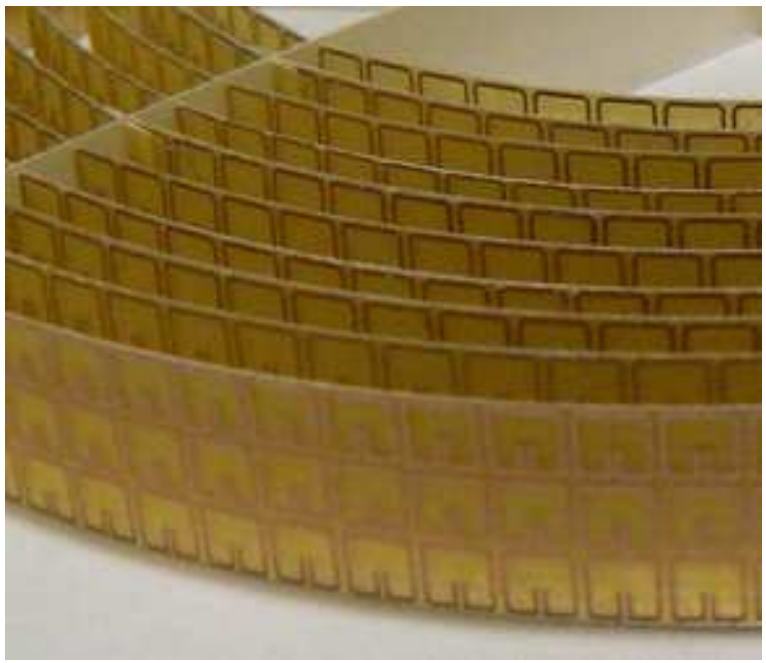
We consider the question: What happens on $\Sigma = \partial B(1)$?

Figure: Invisibility cloak for 4 cm waves build using metamaterials, Schurig et al, Science 2006.



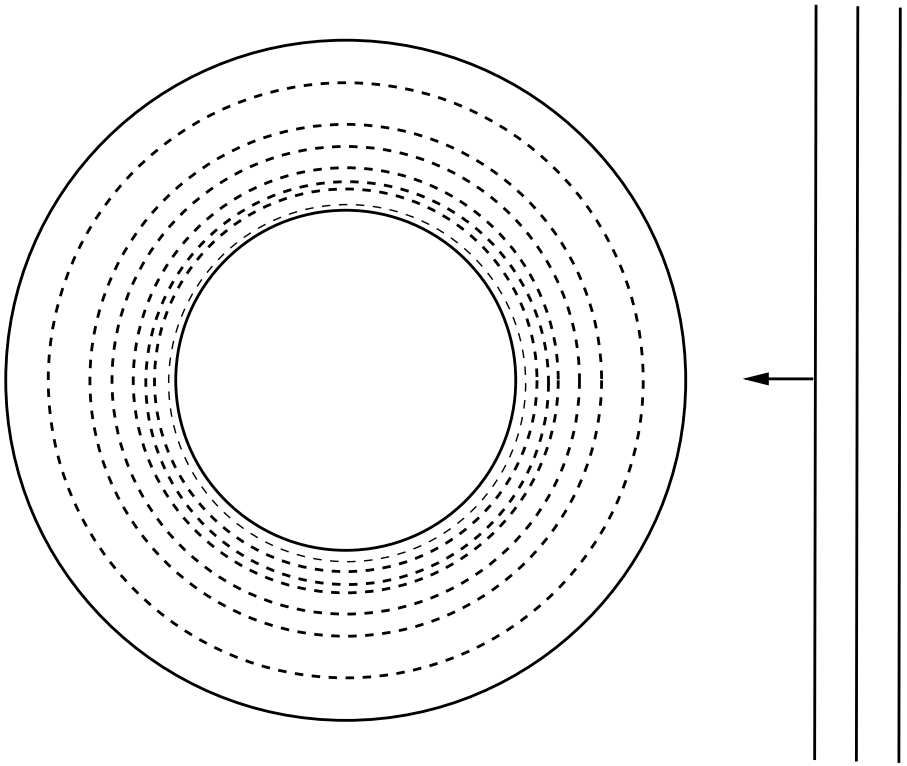
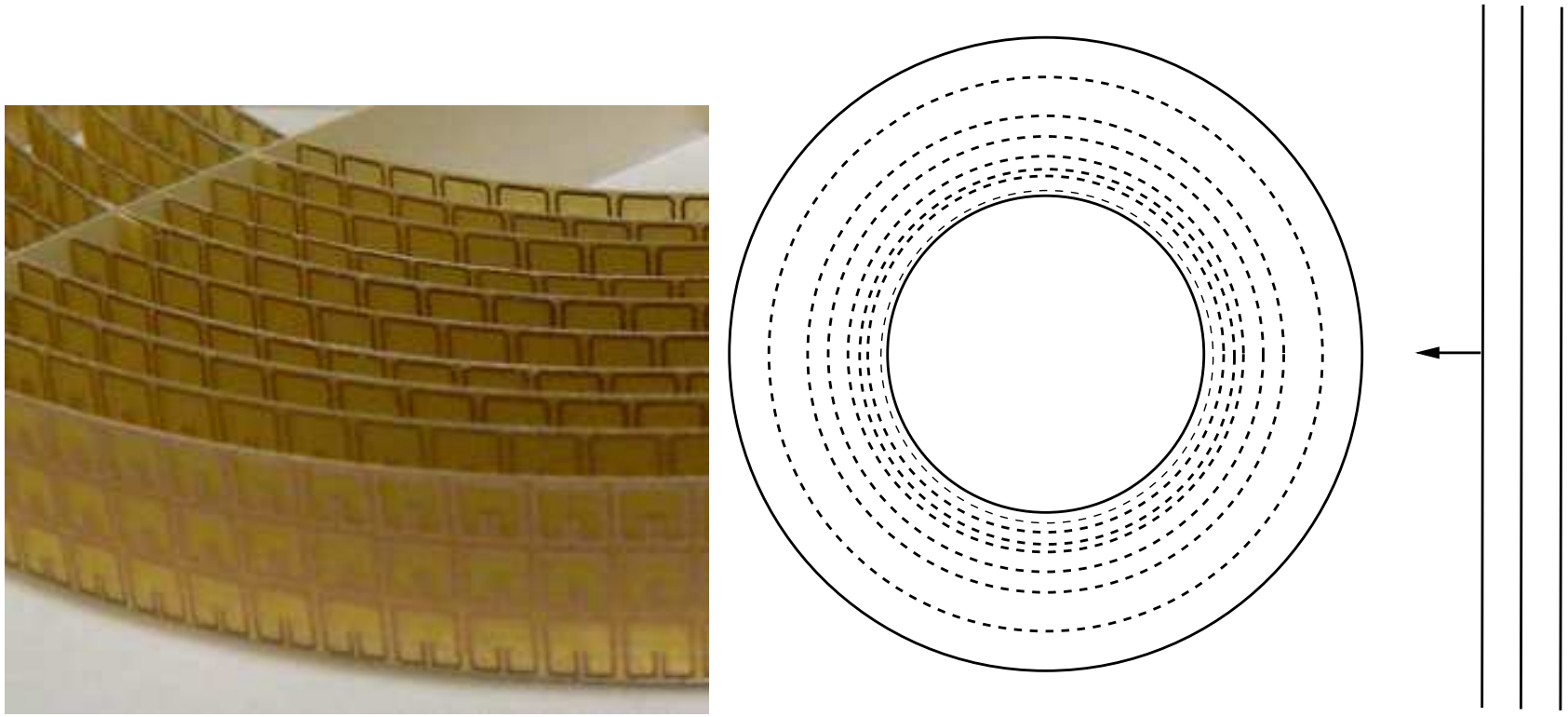
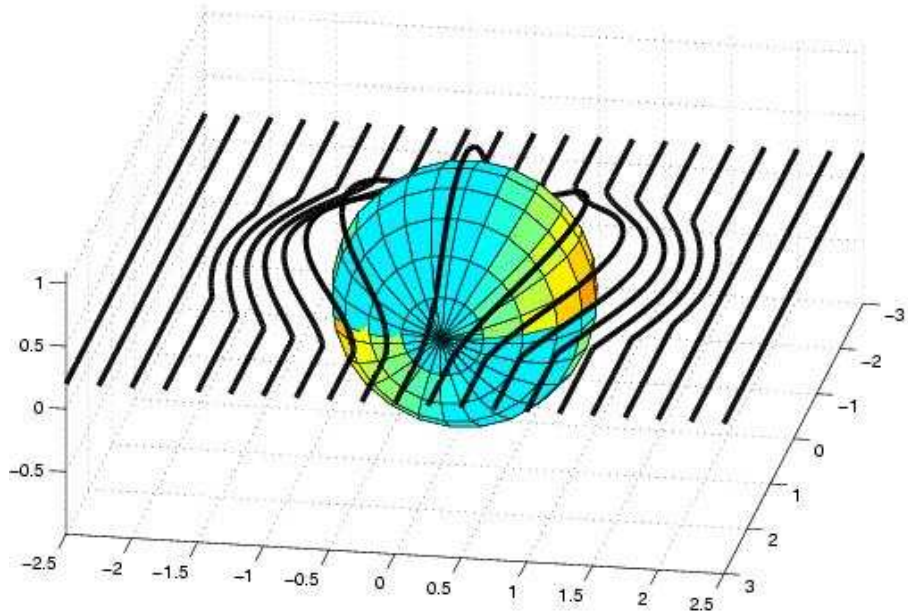


Figure: Invisibility cloak for 4 cm waves build using metamaterials, Schurig et al, Science 2006.



The presently built materials work with a single frequency. There are theoretical obstructions showing that the materials can not work with all frequencies.



“Invisibility in rays” is shown in figure.

For “Invisibility in waves” we need to analyze solutions near the cloaking surface $\partial B(1)$.
The wave have to be quite regular functions E and H as the homogenization theory concerns regular effective fields.

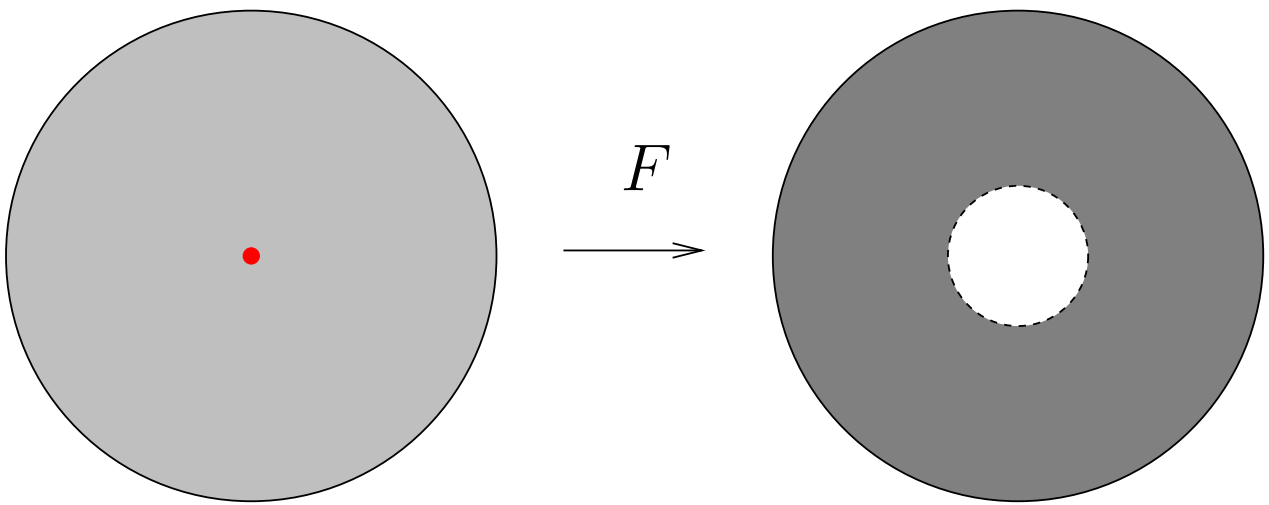
Helmholtz equation. Let $g_{jk} = \delta_{jk}$ and define

$$\widetilde{g}(x) = \begin{cases} F_* g & \text{for } x \in B(2) \setminus \overline{B}(1) \\ \delta_{jk} & \text{for } x \in B(1). \end{cases}$$

This metric is singular on $\Sigma = \partial B(1)$. We consider

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = 0 \quad \text{on } B(2). \tag{2}$$

The equation (2) can be written as an acoustic equation $\nabla \cdot \widetilde{\sigma} \nabla u + k^2 |\widetilde{g}|^{1/2} u = 0$.



Definition 3.4 *A measurable function* $\widetilde{u}$ *on* $N = B(2)$ *is a finite energy solution of the Helmholtz equation,*

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = \widetilde{f} \quad \text{on } N,$$

if

$$\widetilde{u} \in L^2(N, |\widetilde{g}|^{1/2}dx),$$
$$\int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u}\,\partial_j\widetilde{u}\,dx < \infty,$$

and, for all $\widetilde{\psi} \in C_0^\infty(N)$*,*

$$\int_N [-(|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u})\partial_j\widetilde{\psi} + k^2\widetilde{u}\widetilde{\psi}|\widetilde{g}|^{1/2}]dx = \int_N \widetilde{f}\,\widetilde{\psi}\,|\widetilde{g}|^{1/2}dx.$$

Theorem 5 *(Greenleaf-Kurylev-L.-Uhlmann 2007) Denote* $\Sigma = \partial B(1)$ *and let* H *be the source supported in* $B(1)$*. Then the following are equivalent:*

1. $\widetilde{u}$ *is a finite energy solution to*

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = H \quad \text{on } B(2).$$

2. $\widetilde{u} \circ F = u_1$ *on* $B(2) \setminus \{0\}$*, where*

$$(\Delta + k^2)u_1 = 0 \quad \text{on } B(2),$$

and $\widetilde{u}|_{B(1)} = u_2$*, where*

$$\begin{aligned} (\Delta + k^2)\widetilde{u}_2 &= H \quad \text{on } B(1), \\ \partial_\nu u_2|_\Sigma &= 0. \end{aligned}$$

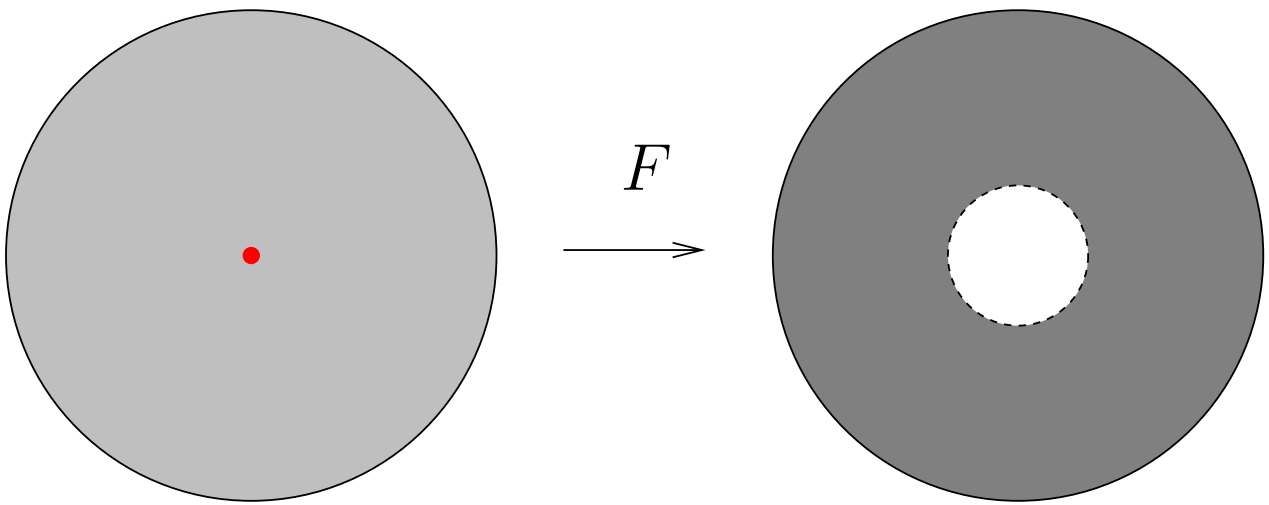
Idea of the proof. Assume that $\widetilde{u}$ is a finite energy solution to

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = H \quad \text{on } B(2).$$

Let $\widetilde{u}^+ = \widetilde{u}|_{B(2)\setminus\overline{B}(1)}$ and $u = \widetilde{u}^+ \circ F$. Then

$$(\Delta + k^2)u = 0 \quad \text{on } B(2) \setminus \{0\}.$$

As capacitance of $\{0\} \subset \mathbb{R}^3$ is zero, we see that the Helmholtz equation on $B(2) \setminus \{0\}$ extends to $B(2)$.



We consider the finite energy solution $\widetilde{u}$ of equation

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = H \quad \text{on } B(2).$$

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Let

$$\widetilde{u}^+ = \widetilde{u}|_{B(2)\setminus\overline{B}(1)}, \qquad \widetilde{u}^- = \widetilde{u}|_{B(1)}.$$

Then, formally speaking,

$$\det(\widetilde{g})^{1/2}\widetilde{g}^{jk}\nu_j\partial_k\widetilde{u}^+|_\Sigma = \det(\widetilde{g})^{1/2}\widetilde{g}^{jk}\nu_j\partial_k\widetilde{u}^-|_\Sigma,$$

and since $\det(\widetilde{g})|_{\Sigma+} = 0$, this would yield $\partial_\nu\widetilde{u}^-|_\Sigma = 0$. Next we consider the Neumann boundary condition rigorously.

We consider the finite energy solution $\widetilde{u}$ of equation

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = H \quad \text{on } B(2).$$

For all $\widetilde{\psi} \in C_0^\infty(B(2))$, $u = \widetilde{u} \circ F$, and $\psi = \widetilde{\psi} \circ F$,

$$\begin{aligned}
& \int_{B(1)} H\widetilde{\psi}\, dx \\
= & \int_{(B(2)\setminus\overline{B}(1))\cup B(1)} [-(|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u})\partial_j\widetilde{\psi} + k^2\widetilde{u}\widetilde{\psi}|\widetilde{g}|^{1/2}]dx \\
= & \int_{B(2)\setminus\{0\}} [-\nabla u\cdot\nabla\psi + k^2 u\psi]dx + \int_{B(1)} [-\nabla\widetilde{u}\cdot\nabla\widetilde{\psi} + k^2\widetilde{u}\widetilde{\psi}]dx \\
= & \; 0 + \int_{B(1)} H\widetilde{\psi}\, dx - \int_{\partial B(1)} \partial_\nu\widetilde{u}\cdot\widetilde{\psi}\, dS.
\end{aligned}$$

This means that we have the Neumann boundary condition on the boundary of $B(1)$.

Cloaking of an infinite cylinder. (Opt. Exp. 2007)
Let $D = \{(r, \theta, z);\ r \leq 1\}$ be an infinite cylinder and cloak it using the map $r \mapsto (1 + r/2)$, $0 < r < 2$. Consider

$$\nabla \times E = i\omega\widetilde{\mu}(x)H, \quad \nabla \times H = -i\omega\widetilde{\epsilon}(x)E + J \quad \text{on } \mathbb{R}^3 \setminus D.$$

- The finite energy solutions do not exist generally.
- The finite energy solutions exists if we put on ∂D a surface with the Soft-and-Hard boundary condition,

$$e_\theta \cdot E|_{\partial D} = 0 \quad \text{and} \quad e_\theta \cdot H|_{\partial D} = 0,$$

where e_θ is the angular unit vector.

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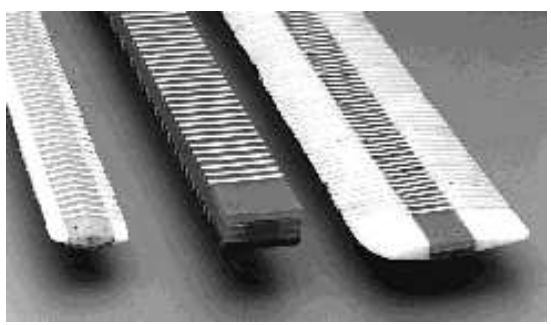
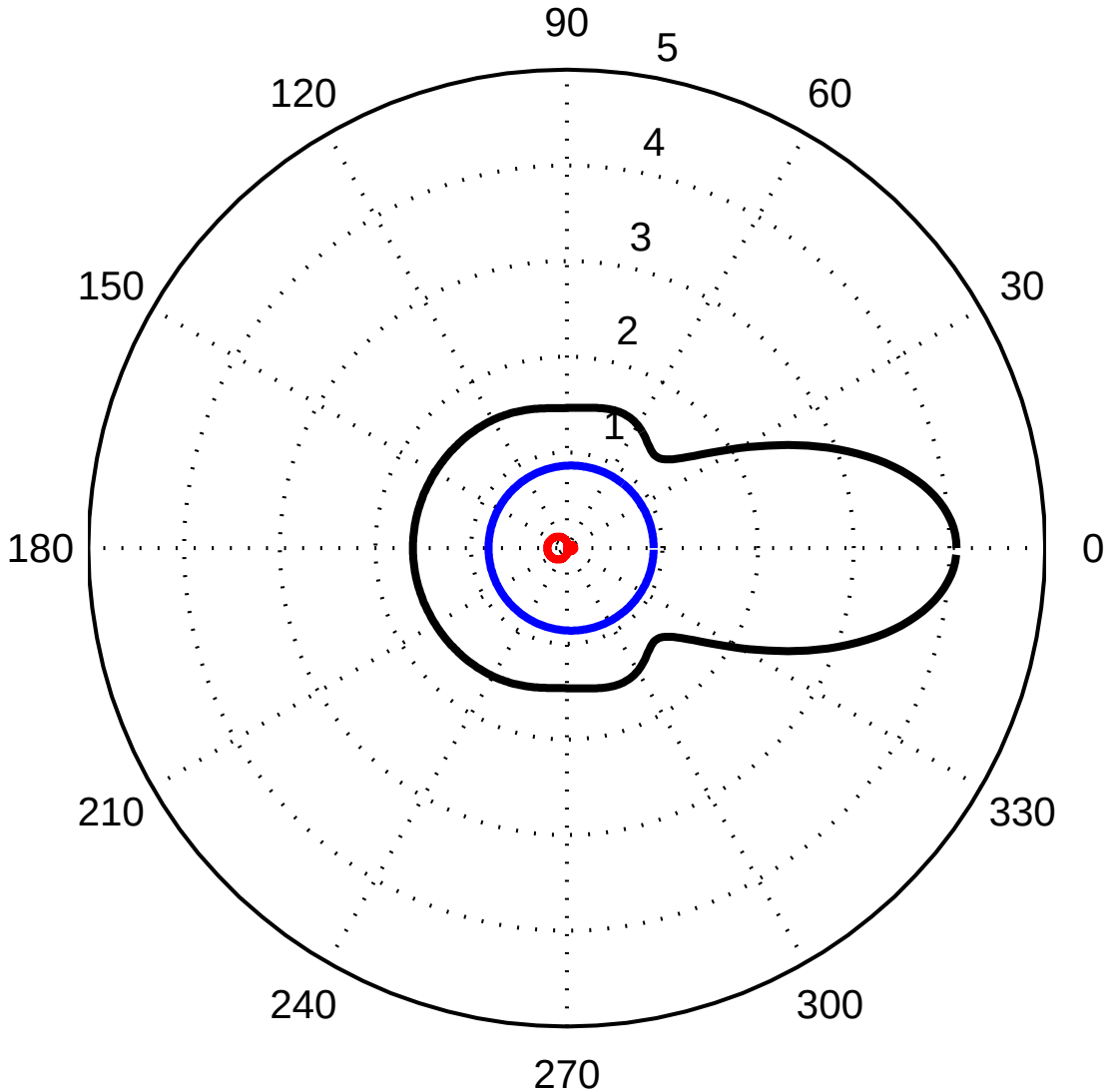


Soft-Hard surfaces created by Kildal et al.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