Detection of thin inclusions in linear elasticity

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Outline

The direct problem

1. Asymptotic formula for displacement field in the presence of small volume inclusions
2. Thin inclusions in an elastic body

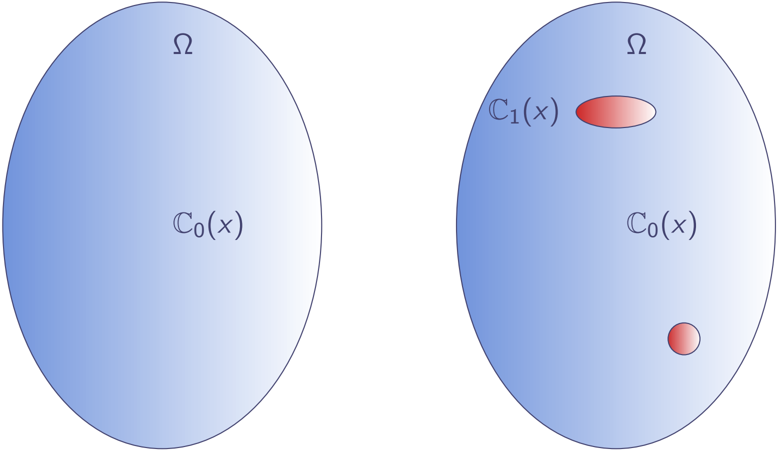
The inverse problem

1. The correction term: a crack model
2. Uniqueness
3. Stability
4. Reconstruction
5. Related problems

Part I

Small volume inclusions

Inclusions in an elastic body



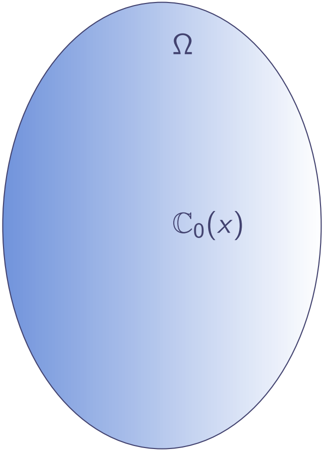
Reference medium

Perturbed medium

Setting:

The volume of the inclusion is small with respect to the volume of Ω

The reference displacement field



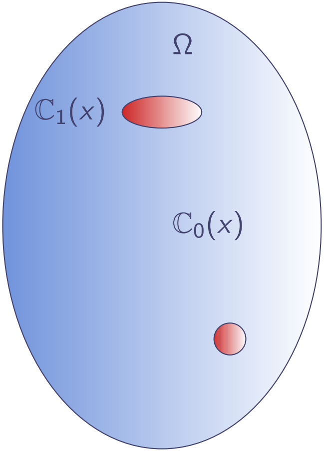
Reference medium

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$ bounded, smooth.
- $\mathbb{C}_0$ be smooth, fully symmetric and strongly convex tensor.
- Background displacement field $u_0 \in \tilde{H}(\Omega, \mathbb{R}^d)$

$$\begin{cases} \operatorname{div}(\mathbb{C}_0 \widehat{\nabla} u_0) &= 0 \text{ in } \Omega \\ (\mathbb{C}_0 \widehat{\nabla} u_0)\nu &= \psi \text{ on } \partial\Omega, \end{cases}$$

where $\psi \in H^{-1/2}(\partial\Omega, \mathbb{R}^d)$ satisfies some compatibility condition

The perturbed displacement field



Perturbed medium

- ω_ϵ denotes the inclusion inside Ω.
- $\mathbb{C}_1$ elasticity tensor inside ω_ϵ, fully symmetric and strongly convex.
- $$\mathbb{C}_\epsilon = \mathbb{C}_0\chi_{\Omega\setminus\overline{\omega}_\epsilon} + \mathbb{C}_1\chi_{\omega_\epsilon},$$
- perturbed displacement field: $u_\epsilon \in \tilde{H}(\Omega, \mathbb{R}^d)$ solution to

$$\begin{cases} \text{div}(\mathbb{C}_\epsilon\widehat{\nabla} u_\epsilon) & = & 0 \text{ in } \Omega \\ (\mathbb{C}_\epsilon\widehat{\nabla} u_\epsilon)\nu & = & \psi \text{ on } \partial\Omega, \end{cases}$$

Assumptions and definitions

On the elasticity tensor

- $\mathbb{C}_0 \in C^{1,\alpha}(\Omega)$, $\mathbb{C}_1 \in C^{\alpha}(\Omega)$ are fully symmetric: for $m = 0, 1$

$$(\mathbb{C}_m(x))_{ijkl} = (\mathbb{C}_m(x))_{klij} = (\mathbb{C}_m(x))_{jikl}, \text{ for } 0 \le i, j, k, l \le d, x \in \Omega.$$

- strongly convex: there is a positive number λ_0 such that

$$\mathbb{C}_m(x)A : A \ge \lambda_0 |A|^2 \text{ for every } d \times d \text{ symmetric matrix } A \text{ and } x \in \Omega.$$

On the solution

- Compatibility condition on the Neumann datum $\psi \in H^{-1/2}(\partial\Omega, \mathbb{R}^d)$, $\int_{\partial\Omega} \psi \cdot R$ for every infinitesimal rigid motion R
- Normalization conditions:

$$\tilde{H}(\Omega, \mathbb{R}^d) = \left\{ u \in H^1(\Omega, \mathbb{R}^d) \; : \; \int_{\partial\Omega} u = 0, \; \int_{\Omega} \nabla u - (\nabla u)^T = 0 \right\}$$

The inclusions

Assume that

- ω_ϵ is measurable,
- $d(\omega_\epsilon, \partial\Omega) \geq d_0 > 0$
- $|\omega_\epsilon| > 0$ and $|\omega_\epsilon| \to 0$ as $\epsilon \to 0$

Strategy:

expand $u_\epsilon - u_0$ with respect to ϵ

Important ingredient

Neumann matrix for the background problem

$$\begin{cases} \operatorname{div}_x \left(\mathbb{C}_0 \hat{\nabla} N(x,y) \right) & = & -\delta_y(x) I_d \quad \text{in} \quad \Omega \\ (\mathbb{C}_0 \hat{\nabla} N)\, \nu & = & -\frac{1}{|\partial\Omega|} I_d \quad \text{su} \quad \partial\Omega, \\ \int_{\partial\Omega} N = 0, & & \int_\Omega (\nabla N - \nabla N^T) = 0. \end{cases}$$

For existence and behaviour of N: M. Fuchs, *Manuscripta Math., 1984.*

If $d = 2$ and $\mathbb{C}_0$ is a homogeneous isotropic elasticity tensor with Lamé constants λ_0, μ_0,

$$\Gamma_{ij}(x,y) := \frac{A}{2\pi} \delta_{ij} \ln|x-y| - \frac{B}{2\pi} \frac{(x-y)_i (x-y)_j}{|x-y|^2},$$

where $A = \frac{1}{2}\left(\frac{1}{\mu_0} + \frac{1}{\lambda_0 + 2\mu_0}\right)$ and $B = \frac{1}{2}\left(\frac{1}{\mu_0} - \frac{1}{\lambda_0 + 2\mu_0}\right)$

The asymptotic formula

Theorem

Let ω_{ϵ_j} be a sequence of measurable subsets of Ω such that $d(\omega_{\epsilon_j}, \partial\Omega) \geq d_0 > 0$ and, for $j \to \infty$,

$$|\omega_{\epsilon_j}| \to 0 \text{ and } |\omega_{\epsilon_j}|^{-1}\chi_{\omega_{\epsilon_j}} dx \to d\mu \text{ in the weak}^* \text{ topology of } (C(\overline{\Omega}))',$$

for some regular positive Borel measure μ, such that $\int_\Omega d\mu = 1$.

There exists a subsequence, not relabeled, and a fourth order tensor $\mathbb{M} \in L^2(\Omega, d\mu)$ such that

$$(u_{\epsilon_j} - u_0)(y) = |\omega_{\epsilon_j}| \int_\Omega \mathbb{M}(x)\widehat{\nabla} u_0(x) : \widehat{\nabla} N(x,y) d\mu_x + o(|\omega_{\epsilon_j}|),$$

for $y \in \partial\Omega$.

[E. Beretta, E. Bonnetier, E.F., A.L. Mazzucato, preprint]

The scalar version of this result is contained in
[Y. CAPDEBOSCQ AND M. VOGELIUS, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, *Math. Modeling Num. Anal., 2003.*]

Some hints

Energy estimate

$$\|u_\epsilon - u_0\|_{H^1(\Omega)} \leq C|\omega_\epsilon|^{1/2}$$

and

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C|\omega_\epsilon|^{1/2+\eta}$$

Correctors

The Elastic Moment Tensor $\mathbb{M}$ is obtained as weak limit of

$$|\omega_\epsilon|^{-1}\chi_{\omega_\epsilon}(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla} v_\epsilon^{ij}$$

where $v_\epsilon^{ij} \in \tilde{H}(\Omega, \mathbb{R}^d)$ are solutions to

$$\begin{cases} \operatorname{div}(\mathbb{C}_\epsilon \widehat{\nabla} v_\epsilon^{ij}) &= \operatorname{div}(\mathbb{C}_0 e_i \otimes e_j) \text{ in } \Omega \\ (\mathbb{C}_\epsilon \widehat{\nabla} v_\epsilon^{ij})\nu &= (\mathbb{C}_0 e_i \otimes e_j)\nu \text{ on } \partial\Omega, \end{cases}$$

Properties of the Elastic Moment Tensor

Proposition

The Elastic Moment Tensor $\mathbb{M}$ *has the same symmetry properties of the elasticity tensors* $\mathbb{C}_0$ *and* $\mathbb{C}_1$*, that is*

$$\mathbb{M}_{ijkl} = \mathbb{M}_{klij} = \mathbb{M}_{jikl}, \qquad \mu\text{-a.e.}$$

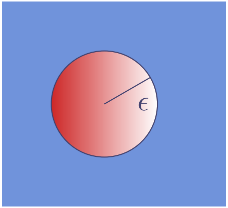
for any choice of indices i, j, k, l *between* 1 *and* d*.*
Moreover, for any symmetric matrix E*,*

$$\mathbb{C}_0\mathbb{C}_1^{-1}(\mathbb{C}_1 - \mathbb{C}_0)E : E \leq \mathbb{M}E : E \leq (\mathbb{C}_1 - \mathbb{C}_0)E : E \qquad \mu\text{-a.e.}.$$

Special geometries: diametrically small inclusions

$\omega_\epsilon = \epsilon B, \quad B$ domain of measure 1

$$|\omega_\epsilon| \approx \epsilon^d$$



References: homogeneous isotropic body

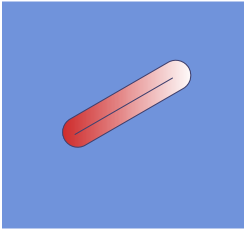
- H. Ammari, H. Kang, G. Nakamura, and K. Tanuma, *Complete asymptotic ...*, J. Elasticity, (2002).
- H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Math. 1846, Springer-Verlag, Berlin, 2004.
- H. Ammari and H. Kang, *Polarization and Moment Tensors: with Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences Series, Volume 162, Springer-Verlag, New York, 2007.

Special geometries: thin inclusions

$$\omega_\epsilon = \{x \in \Omega \,:\, d(x,\sigma) \leq \epsilon\},$$

for σ smooth non intersecting open curve in $\Omega \subset \mathbb{R}^2$

$$|\omega_\epsilon| \approx \epsilon$$



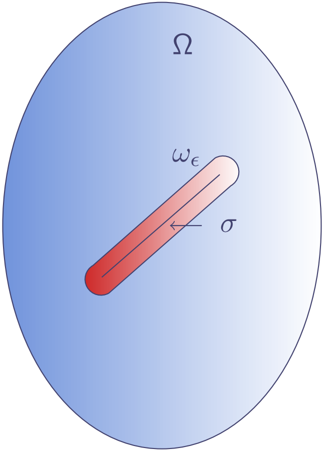
References

- (Isotropic inclusions) E. Beretta and E. F., *An asymptotic formula for the displacement field in the presence of thin elastic inhomogeneities*, SIAM J. Math. Anal., 38 (2006), 1249-1261.
- E. Beretta, E. Bonnetier, E. F. and A. L. Mazzucato, (2011) preprint.

Part II

Thin inclusions

Thin inclusions in a plane elastic body: isotropic homogeneous



$\Omega \subset \mathbb{R}^2$ is a plane region occupied by an elastic material containing an inclusion of the form

$$\omega_\epsilon = \{x \in \Omega \,:\, d(x, \sigma) < \epsilon\}$$

where σ is a line segment.

$$\mathbb{C}_\epsilon = \mathbb{C}_0 \chi_{\Omega \setminus \overline{\omega}_\epsilon} + \mathbb{C}_1 \chi_{\omega_\epsilon}$$

$$\{\mathbb{C}_m\}_{ijkl=1}^{2} = \lambda_m \delta_{ij} \delta_{kl} + \mu_m (\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li})$$

Asymptotic formula

Theorem

For every $y \in \partial\Omega$ and for $\epsilon \to 0$

$$(u_\epsilon - u_0)(y) = 2\epsilon \int_\sigma \mathbb{M}(x)\widehat{\nabla} u_0(x) : \widehat{\nabla} N(x,y)\, d\sigma(x) + o(\epsilon),$$

where

$$\mathbb{M}\widehat{\nabla} u_0 = a\, div\, u_0\, I_d + b\widehat{\nabla} u_0 + c\frac{\partial(u_0\cdot\tau)}{\partial\tau}\tau\otimes\tau + d\frac{\partial(u_0\cdot n)}{\partial n}n\otimes n,$$

where τ and n are the tangent and normal direction to segment σ and a, b, c, d, e and f depend only on the Lamé coefficients of $\mathbb{C}_0$ and $\mathbb{C}_1$. The term $o(\epsilon)$ is bounded by $C\epsilon^{1+\theta}\|\psi\|_{H^{-1/2}(\partial\Omega)}$, with $0 < \theta < 1$ and C depending only on Ω, α_0, β_0 and K.

[E. Beretta and E. F., SIAM J. Math. Anal., 2006.]

The solution u_ϵ

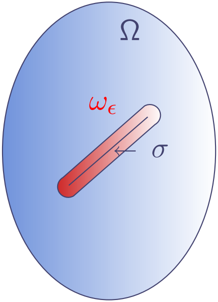
$$\text{Set } u_\epsilon^e := u_{\epsilon|_{\Omega\setminus\overline{\omega}_\epsilon}} \text{ and } u_\epsilon^i := u_{\epsilon|_{\omega_\epsilon}}.$$

$$u_\epsilon^e \in C^{1,\alpha}(\Omega \setminus \omega_\epsilon) \text{ and } u_\epsilon^i \in C^{1,\alpha}(\overline{\omega}_\epsilon).$$

[Y.Y.Li and L. Nirenberg, Comm. Pure Appl. Math., 2003]

Equations:

$$\begin{aligned} \operatorname{div}(\mathbb{C}_0 \widehat{\nabla} u_\epsilon^e) &= 0 \quad \text{in} \quad \Omega \setminus \overline{\omega}_\epsilon, \\ \operatorname{div}(\mathbb{C}_1 \widehat{\nabla} u_\epsilon^i) &= 0 \quad \text{in} \quad \omega_\epsilon. \end{aligned}$$



Transmission conditions:

$$u_\epsilon^e = u_\epsilon^i \quad \text{su} \quad \partial\omega_\epsilon,$$

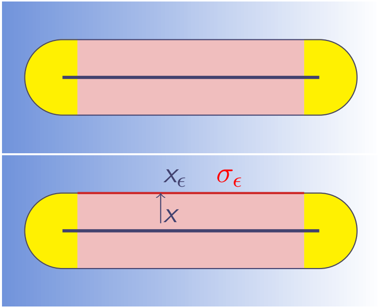
$$(\mathbb{C}_0 \widehat{\nabla} u_\epsilon^e)\, n = (\mathbb{C}_1 \widehat{\nabla} u_\epsilon^i)\, n \quad \text{su} \quad \partial\omega_\epsilon.$$

where n is the outer normal to $\partial\omega_\epsilon$,

Proof's main steps: localization to the boundary of the inclusion

- First, we show that $y \in \partial\Omega$,

$$(u_\epsilon - u_0)(y) = \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla} u^i_\epsilon(x) : \widehat{\nabla} N(x,y)\, dx.$$



- The yellow part of ω_ϵ gives a contribution of order $\epsilon^{1+\beta}$.

- $$\nabla u^i_\epsilon \in C^\alpha(\overline{\omega}^*_\epsilon)$$
$$\widehat{\nabla} u^i_\epsilon(x) \approx \widehat{\nabla} u^i_\epsilon(x_\epsilon)$$

- $$(u_\epsilon - u_0)(y) = 2\epsilon \int_{\sigma_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla} u^i_\epsilon(x_\epsilon) : \widehat{\nabla} N(x_\epsilon, y)\, d\sigma_\epsilon + o(\epsilon).$$

Proof's main steps: outside the inclusion and final

$$(u_\epsilon - u_0)(y) = 2\epsilon \int_{\sigma_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla} u^i_\epsilon(x_\epsilon) : \widehat{\nabla} N(x_\epsilon, y)\, d\sigma_\epsilon + o(\epsilon).$$

We show that, there exists a tensor $\mathbb{M}$ such that

$$(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla} u^i_\epsilon(x_\epsilon) = \mathbb{M}\widehat{\nabla} u^e_\epsilon$$

In the isotropic case, the existence of such a tensor $\mathbb{M}$ follows immediately by the transmission conditions by solving a linear system.

$$(u_\epsilon - u_0)(y) = 2\epsilon \int_{\sigma_\epsilon} \mathbb{M}\widehat{\nabla} u^e_\epsilon(x_\epsilon) : \widehat{\nabla} N(x_\epsilon, y)\, d\sigma_\epsilon + o(\epsilon).$$

Finally we show that

$$\|\nabla u^e_\epsilon - \nabla u_0\|_{L^\infty(\sigma_\epsilon)} \to 0$$

and get

$$(u_\epsilon - u_0)(y) = 2\epsilon \int_{\sigma} \mathbb{M}(x)\widehat{\nabla} u_0(x) : \widehat{\nabla} N(x, y)\, d\sigma(x) + o(\epsilon),$$

Remark

An analogue asymptotic expansion holds true if σ is a regular curve or a collection of multiple disjoint curves.

More general elasticity tensor

This asymptotic expansion holds true if we can establish:

- Existence and local estimates for the Neumann matrix for background problem

  [M. Fuchs, 1984]

- Interior regularity estimates for solutions.

  [S. Campanato, Quaderni della SNS di Pisa, 1980.]

  Caccioppoli and Meyer's type inequalities for strongly convex systems with L^∞ coefficients.

- Existence of the Elastic Moment tensor such that

$$(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla} u_\epsilon^i = \mathbb{M}\widehat{\nabla} u_\epsilon^e$$

[G. A. Francfort and F. Murat, *Homogenization and Optimal Bounds in Linear Elasticity, Arch. Rational Mech. Anal., 1986.*]

Asymptotic expansion

Theorem

Let $\mathbb{C}_0 \in C^{1,\alpha}(\Omega)$ and $\mathbb{C}_1 \in C^1(\Omega)$ be fully symmetric and strongly convex tensors. For every $y \in \partial\Omega$ and for $\epsilon \to 0$

$$(u_\epsilon - u_0)(y) = 2\epsilon \int_\sigma \mathbb{M}(x)\widehat{\nabla} u_0(x) : \widehat{\nabla} N(x,y)\, d\sigma(x) + o(\epsilon),$$

where, for any symmetric matrix H,

$$\mathbb{M}H = (\mathbb{C}_0 - \mathbb{C}_1)H + (\mathbb{C}_0 - \mathbb{C}_1)\left(q\left((\mathbb{C}_0 - \mathbb{C}_1)Hn\right) \otimes n\right).$$

Here q denotes the inverse of

$$(q^{-1}\zeta)\zeta = \mathbb{C}_1(\zeta \otimes n) : (\zeta \otimes n)$$

[E. Beretta. E. Bonnetier, E. F., A.L. Mazzucato, preprint.]

Part III

The inverse problem

References

Thin inclusions: isotropic bodies

- E. Beretta, E. F and S. Vessella *Determination of a linear crack in an elastic body from boundary measurements - Lipschitz Stability*, SIAM J. Math. Anal., 2006.
- E. Beretta, E. F., E. Kim and J.-Y. Lee *Algorithm for the determination of a linear crack in an elastic body from boundary measurements*, Inverse Problems, 2010.

The correction term

The asymptotic expansion gives:

$$(u_\epsilon - u_0)(y) = \epsilon u_\sigma(y) + o(\epsilon).$$

where

$$u_\sigma(y) = 2\int_\sigma \mathbb{M}(x)\widehat{\nabla} u_0(x) : \widehat{\nabla} N(x,y)\, d\sigma(x)$$

is the first order approximation of u_ϵ that we will call correction term.
From now on we consider homogeneous isotropic strongly convex tensors with:

$$\mu_l \geq \alpha_0 > 0, \quad \lambda_l + \mu_l \geq \beta_0 > 0 \quad \text{for} \quad l = 0, 1.$$

We also assume,

$$(\lambda_0 - \lambda_1)^2 + (\mu_0 - \mu_1)^2 > 0 \quad \text{and} \quad (\lambda_0 - \lambda_1)(\mu_0 - \mu_1) \geq 0.$$

The correction term

Consider a linear background

$$u_0(x) = Wx + c,$$

where W is a non zero symmetric matrix.
The function u_σ con be expressed in the following way

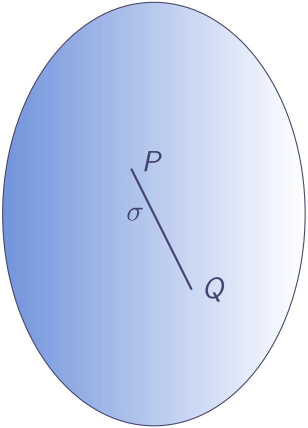
$$u_\sigma(y) = \int_\sigma \mathbb{C}_0 \widehat{\nabla} N(x,y) n \cdot \varphi \, d\sigma(x) + ((N(Q,y)\cdot\tau) - (N(P,y)\cdot\tau)) \, f.$$

Here

- P and Q are the endpoints of segment σ,
- $\tau = \frac{PQ}{|PQ|}$ and $n = \tau^\perp$,
- φ is a vector valued function, f is a constant and they both depend only on W, on τ and n and on the Lamé coefficients.

The crack model

The correction term u_σ is the trace of a function that has the following properties:



- In $\Omega \setminus \sigma$ it solves the background system

$$\text{div}\left(\mathbb{C}_0 \hat{\nabla} u_\sigma\right) = 0.$$

- At the endpoints $\{P, Q\}$ of the segment σ it has singularities proportional to $N(Q, y) \cdot \tau$ and $N(P, y) \cdot \tau$.
- It jumps across the segment σ.
- The correction term u_σ has a zero conormal derivative on $\partial\Omega$.

Inverse problem

Problem:

Given the trace of the correction term on an open subset of $\partial\Omega$, determine the segment σ.

Uniqueness

Proposition

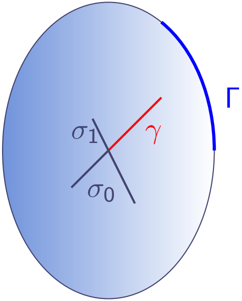
Let σ_0 and σ_1 be two segments contained in the interior of the open set Ω. Let u_{σ_0} and u_{σ_1} be the correction terms corresponding to $\psi = (\mathbb{C}_0 W)\nu$. Let Γ be an open subset of $\partial\Omega$. If

$$u_{\sigma_0} = u_{\sigma_1} \quad \text{on} \quad \Gamma,$$

then

$$\sigma_0 = \sigma_1.$$

Sketch of proof



By unique continuation, since their Cauchy data coincide on Γ,

$$u_{\sigma_0} = u_{\sigma_1} \quad \text{on} \quad \Omega \setminus (\sigma_0 \cup \sigma_1.)$$

If $\sigma_0 \neq \sigma_1$, there is a part γ of σ_0 not contained in σ_1.
$\Rightarrow$ u_{σ_0} is bounded and has no jumps on γ.

$$u_{\sigma_0}(y) = \int_{\sigma_0} \mathbb{C}_0 \widehat{\nabla} N(x,y) n \cdot \varphi_0 \, d\sigma(x) + ((N(Q,y) \cdot \tau) - (N(P,y) \cdot \tau)) f_0.$$

But $\varphi_0 = 0$ and $f_0 = 0$ imply $W = 0$.

Lipschitz Stability

Theorem

Let σ_0 and σ_1 be two segments contained in Ω, far from the boundary and of positive length. Let u_{σ_0} and u_{σ_1} be the correction terms corresponding to $\psi = (\mathbb{C}_0 W)\nu$, and let Γ be an open subset of $\partial\Omega$.
There exists a constant C depending only on the a priori data such that

$$d_{\mathcal{H}}(\sigma_0, \sigma_1) \leq C \| u_{\sigma_0} - u_{\sigma_1} \|_{L^2(\Gamma)}.$$

where $d_{\mathcal{H}}$ denotes the Hausdorff distance.

[E.Beretta, E. F., S. Vessella, SIAM J. Math. Anal.,2008]

Idea of the proof

Fix $u_0 = Wx + c$ non identically zero.

Let us define $T : \Lambda \subset \mathbb{R}^4 \to H^{1/2}(\Gamma)$ as the operator that associate to a segment σ the trace on Γ of u_σ.

We prove that, if we take K a compact subset of Λ, then:

- T is injective (uniqueness).
- T is Frèchet differentiable and its derivative is continuous and injective.

This implies that T^{-1} is continuous, hence, there is a constant C such that, taken two segments $\sigma_0, \sigma_1 \in K$, then

$$d_{\mathcal{H}}(\sigma_0, \sigma_1) \leq C\|u_{\sigma_0} - u_{\sigma_1}\|_{L^2(\Gamma)}.$$

Rescaled stability estimate

Corollary

Let σ_0 and σ_1 be two segments contained in Ω, far from the boundary and of positive length and let Γ be an open subset of $\partial\Omega$.
For $j = 0, 1$, let u_ϵ^j be the solution of

$$\begin{cases} div\left(\mathbb{C}_\epsilon \hat{\nabla} u_\epsilon^j\right) = 0 & in \quad \Omega \\ (\mathbb{C}_\epsilon \hat{\nabla} u_\epsilon^j)\, \nu = g & on \quad \partial\Omega, \end{cases}$$

corresponding to $\omega_\epsilon^j = \{x \in \Omega \,:\, d(x, \sigma_j) < \epsilon\}$.
Then, there exist a positive constant C and $\theta \in (0, 1)$, depending only on the a priori data, such that,

$$d_{\mathcal{H}}(\sigma_0, \sigma_1) \leq C\left(\epsilon^{-1}\|u_\epsilon^0 - u_\epsilon^1\|_{L^2(\Gamma)} + \epsilon^\theta\right).$$

Reconstruction: MELIR algorithm

[E. Beretta, E. F., E. Kim and J.-Y. Lee Inverse Problems, 2010]

$$u_\sigma(y) = \int_\sigma \mathbb{C}_0 \widehat{\nabla} N(x,y) n \cdot \varphi \, d\sigma(x) + ((N(Q,y)\cdot\tau) - (N(P,y)\cdot\tau))\, f.$$

Idea:
there is a symmetric matrix W such that, for $\psi = (\mathbb{C}_0 W)\nu$,

$$u_\sigma^W(y) = (N(y,Q) - N(y,P)) \cdot \frac{Q-P}{|Q-P|}.$$

$$(N(y,Q) - N(y,P)) \cdot \frac{Q-P}{|Q-P|} \text{ is in the range of } W \to u_\sigma^W$$

Algorithm

Discretize:
We take Ω to be the unit disk centered at zero and choose N equi-spaced points y_i along the boundary $\partial\Omega$. Choose W_j, $j = 1, 2, 3$ a basis in the space of symmetric matrices, $\psi_j = (\mathbb{C}_0 W_j)\nu$.
Let $A := (u_\sigma^{W_j}(y_i))$ and consider its spectral decomposition

$$A = \sum_{p=1}^{3} s^p u^p \otimes v^p.$$

Let $\mathbb{P} : \mathbb{R}^N \longrightarrow span\{u^1, u^2, u^3\}$ be the orthogonal projector $\mathbb{P} = \sum_{p=1}^{3} u^p \otimes u^p$.

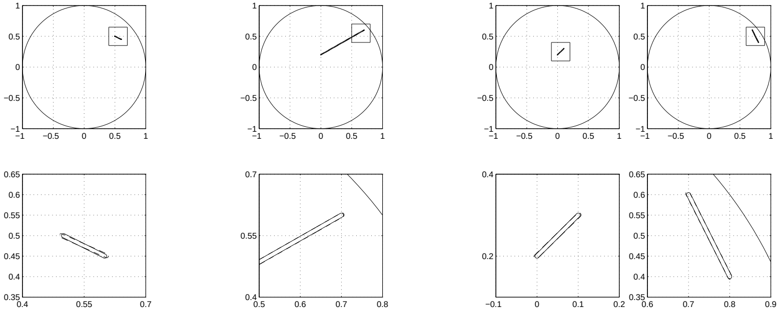
$$(I - \mathbb{P})(N(Q, y_i) - N(P, y_i)) \cdot \frac{Q - P}{|Q - P|} = 0.$$

Algorithm

Note that we cannot measure u_σ^W but $(u_\epsilon^W - u_0^W) = \epsilon u_\sigma^W + o(\epsilon)$

- Let $A_\epsilon := ((u_\epsilon^{W_j} - u_0^{W_j})(y_i))$. The $2N \times 3$ matrix A_ϵ has a spectral decomposition
$$A_\epsilon = \sum_{p=1}^{3} s_\epsilon^p u_\epsilon^p \otimes v_\epsilon^p,$$
- Let $\mathbb{P}_\epsilon : \mathbb{R}^N \longrightarrow span\{u_\epsilon^1, u_\epsilon^2, u_\epsilon^3\}$ be the orthogonal projector $\mathbb{P}_\epsilon = \sum_{p=1}^{3} u_\epsilon^p \otimes u_\epsilon^p$.
- Compute P^c and Q^c that minimize
$$L(P,Q) := \frac{\|(I - \mathbb{P}_\epsilon)((N(Q,y_i) - N(P,y_i)) \cdot \tau)\|_{L^2(\partial\Omega)}}{\|\mathbb{P}_\epsilon((N(Q,y_i) - N(P,y_i)) \cdot \tau)\|_{L^2(\partial\Omega)}}$$
- Compute
$$\epsilon^c = \frac{\|u_\epsilon^W - u_0^W\|_{L^2(\partial\Omega)}}{\|u_{\sigma^c}^W\|_{L^2(\partial\Omega)}}.$$

Numerical experiments: MELIR algorithm

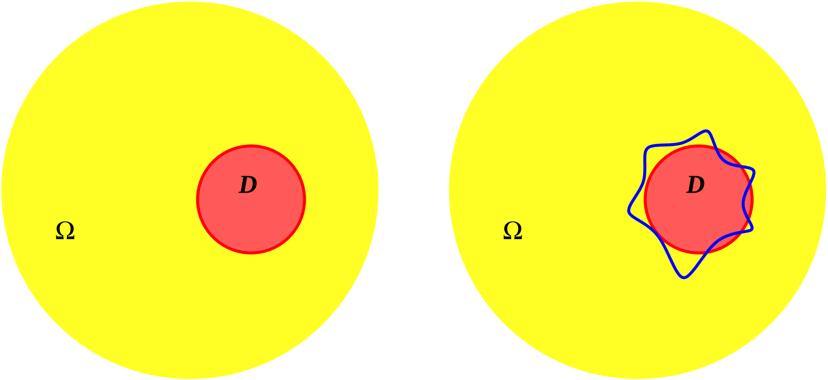


Case	P	Q	ϵ	$\frac{\|\|P-P^c\|\|}{\|\|P-Q\|\|}$	$\frac{\|\|Q-Q^c\|\|}{\|\|P-Q\|\|}$	$\frac{\|\epsilon^c-\epsilon\|}{\epsilon}$	Err
1	(0.5,0.5)	(0.6,0.45)	5.0e-3	0.050	0.042	0.070	0.0964
2	(0.0,0.2)	(0.7,0.6)	5.0e-3	0.002	0.002	0.057	0.0574
3	(0.0,0.2)	(0.1,0.3)	5.0e-3	0.013	0.013	0.053	0.0558
4	(0.7,0.6)	(0.8,0.4)	5.0e-3	0.013	0.009	0.008	0.0178

Part IV

Small interface changes

Small perturbation of an interface



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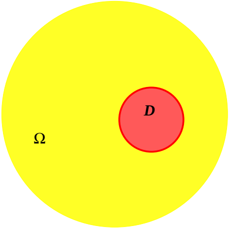
H. Ammari, E. Beretta, E. F., H. Kang, M. Lim, *Journal de Mathématiques Pures et Appliquées*, 2010.

Small perturbations of the elastic interface

$\Omega \subset \mathbb{R}^2$ is a planar region occupied by an elastic body containing an inclusion D of a different material.

Let $\mathbb{C}_0$ and $\mathbb{C}_1$ be the elastic tensor fields in $\Omega \setminus \overline{D}$ and D respectively.



We assume that both materials are isotropic and homogeneous with Lamé constants (λ_0, μ_0) and (λ_1, μ_1).
Let u_0 be the solution of the following eigenvalue problem:

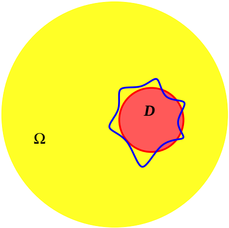
$$\begin{cases} \nabla \cdot \left(\mathbb{C}_D \widehat{\nabla} u_0 \right) = -\omega_0^2 u_0 & \text{in} \quad \Omega, \\ u_0 = 0 & \text{on} \quad \partial\Omega, \\ ||u_0||_{L^2(\Omega)} = 1, & \end{cases}$$

where $\mathbb{C}_D = \mathbb{C}_0 \mathbf{1}_{\Omega \setminus D} + \mathbb{C}_1 \mathbf{1}_D$.

The perturbed problem

Let

$$\partial D_\epsilon = \left\{ x + \epsilon h(x)\nu(x), x \in \partial D \right\},$$



$\nu(x)$ unit outer normal vector to ∂D at x, h smooth function and ϵ positive small parameter.

$$\mathbb{C}_{D_\epsilon} = \mathbb{C}_0 \mathbf{1}_{\Omega \setminus D_\epsilon} + \mathbb{C}_1 \mathbf{1}_{D_\epsilon}$$

Let $u_\epsilon \in H^1(\Omega)$ be the solution to

$$\begin{cases} \nabla \cdot \left(\mathbb{C}_{D_\epsilon} \widehat{\nabla} u_\epsilon \right) & = & -\omega_\epsilon^2 u_\epsilon \quad \text{in} \quad \Omega, \\ u_\epsilon & = & 0 \quad \text{on} \quad \partial\Omega, \\ ||u_\epsilon||_{L^2(\Omega)} & = & 1. \end{cases}$$

Then u_ϵ satisfies the transmission conditions across ∂D_ϵ.

Assumptions

- $d(D, \partial\Omega) \geq K$
- $\partial\Omega \in C^{1,1}$ and $\partial D \in C^{2,1}$
- $h \in C^{1,1}$ and $\|h\|_{C^{1,1}} \leq H$
- Strong convexity of $\mathbb{C}_0$ and $\mathbb{C}_1$:

$$\min(\mu_0, \mu_1) \geq \alpha_0 > 0, \quad \min(2\lambda_0 + 2\mu_0, 2\lambda_1 + 2\mu_1) \geq \beta_0 > 0,$$

Asymptotic expansion of eigenvalue perturbation

Theorem

Let ω_0^2 be a simple eigenvalue. Then, there exists ω_ϵ^2, such that $\omega_\epsilon^2 \to \omega_0^2$ as $\epsilon \to 0$ and

$$\omega_\epsilon^2 - \omega_0^2 = \epsilon \int_{\partial D} h(x) \mathbb{M} \widehat{\nabla} u_0^e(x) : \widehat{\nabla} u_0^e(x) \, d\sigma(x) + O(\epsilon^{1+\beta}),$$

for some positive β and where

$$\mathbb{M} \widehat{\nabla} {u_0}^e = a \operatorname{div} u_0^e \, \mathrm{I}_d + b \widehat{\nabla} u_0 + c \frac{\partial({u_0}^e \cdot \tau)}{\partial \tau} \tau \otimes \tau + d \frac{\partial({u_0}^e \cdot n)}{\partial n} n \otimes n$$

where ν, τ are respectively the outward normal vector and the tangent vector to ∂D and a, b, c, d are constants depending on the Lamé coefficients .

Proof

- Osborn result on convergence of eigenvalues of sequences of self-adjoint collectively compact operators.
- Gradient estimates for solutions of elliptic systems (LI AND NIRENBERG and Meyer's type inequality for L^∞ strongly convex tensors.)

$$\omega_\epsilon^2 - \omega_0^2 = \epsilon \int_{\partial D} h(x)\mathbb{M}\widehat{\nabla} u_0^e(x) : \widehat{\nabla} u_0^e(x)\, d\sigma(x) + O(\epsilon^{1+\beta}),$$

Problem:

From knowledge of eigenvalues and boundary value of eigenfunctions we want to determine the perturbation ϵh

Dual asymptotic formula

Let u_0 be an eigenfunction of the unperturbed problem corresponding to ω_0^2.
For $g \in L^2(\partial\Omega)$ such that $\int_{\partial\Omega} g \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0$, let w_g be the solution to

$$\left\{ \begin{array}{rcll} \nabla \cdot \left(\mathbb{C}_D \widehat{\nabla} w_g\right) & = & -\omega_0^2 w_g & \text{in} \quad \Omega, \\ w_g & = & g & \text{on} \quad \partial\Omega. \end{array} \right.$$

Multiplying the equation for w_g by u_ϵ, by using the divergence theorem, we get

$$\int_{\partial\Omega} g \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_\epsilon^2 - \omega_0^2) \int_\Omega w_g \cdot u_\epsilon = \int_\Omega (\mathbb{C}_{D_\epsilon} - \mathbb{C}_D)\widehat{\nabla} u_\epsilon : \widehat{\nabla} w_g .$$

Dual asymptotic formula

We now use

- Gradient estimates for u_ϵ and w_g
- The asymptotic expansion of the eigenvalues

$$\omega_\epsilon^2 - \omega_0^2 = O(\epsilon)$$

- L^2 estimates for $u_\epsilon - u_0$

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C\epsilon^{\frac{1}{2}+\eta}$$

  for some positive η
- Energy estimates and transmission conditions

Dual asymptotic formula

We derive

Theorem

$$\int_{\partial\Omega} g \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_\epsilon^2 - \omega_0^2) \int_\Omega w_g \cdot u_0$$
$$= \epsilon \int_{\partial D} h(x)\mathbb{M}\widehat{\nabla} u_0^e(x) : \widehat{\nabla} w_g^e(x) d\sigma(x) + O(\epsilon^{1+\beta})$$

for some $\beta > 0$.