Topics course "Complex Dynamics and the Mandelbrot set"

Nataliya Goncharuk, natasha_goncharuk@tamu.edu

Course description:

The Mandelbrot set has attracted attention of the mathematicians for several decades, both because it is beautiful and because it helps visualize effects that appear in many other dynamical systems.

In this course, we will enjoy the beauty of the Mandelbrot set and the Julia sets in the mathematical way: we will learn how to understand it.

The course will cover basic dynamical systems results: linearization at fixed and periodic points (including attracting, super-attracting, repelling, parabolic, and irrational neutral points). Also, we will discuss the technique of the quasiconformal mappings, which is extensively used in the one-dimensional complex dynamics. We will apply these results to understand the geometry of the Julia set and the Mandelbrot set.

Learning outcomes:

Upon successful conclusion of this course students should be able to:

- Know the types of fixed and periodic points of a dynamical system; construct and use linearizing coordinates of fixed points of analytic functions, be familiar with linearization theorems for circle maps;
- Be familiar with basic properties of quasiconformal mappings and the Ahlfors-Bers theorem.
- Understand the underlying dynamics for computer-generated pictures of Julia sets; understand the meaning of the Mandelbrot set and its parts for the dynamics of quadratic polynomials;

- Know the basic definitions and theorems in the dynamics of the quadratic polynomials.

Grading scheme [tentative]: 30% attendance and class participation + 50% bi-weekly homework assignments + 20% take-home final test.
Prerequisites: any course in Complex Analysis.
Literature:

- J.Milnor, "One-dimensional complex dynamics";
- L.Carleson, T.Gamelin, "Complex dynamics".

Tentative schedule:

- Week 1. Introduction to dynamical systems: periodic points, dense orbits, chaotic behavior.
- Week 2. Normal families and Montel's theorem. Definition of the Fatou set and the Julia set for rational maps in $\mathbb{C}$.
- Week 3. Linearizing coordinates for attracting, super-attracting, and repelling periodic points. Cremer points and Siegel discs.
- Week 4. Quasiconformal maps, Ahlfors-Bers theorem.
- Week 5. Parabolic points, Fatou flowers. Uniformization for parabolic points (using quasiconformal maps).
- Week 6. Repelling orbits are dense in the Julia set of a rational map.
- Week 7. Orbits of critical points. Julia sets for polynomials: Bottcher coordinate at infinity, connectedness vs total disconnectedness of Julia sets. Cantor dusts.
- Week 8. Sullivan classification of Fatou components (statement). Hyperbolic components of the Mandelbrot set and related Julia sets (basilica, cauliflower, rabbit etc).
- Week 9. Roots of the limbs of the Mandelbrot set (fat rabbit, fat basilica etc). Absence of continuous dependence for the Julia sets; Douady elephants.

- Week 10. Herman rings; dynamics and linearization of circle maps. Julia sets of Blachke products.
- Week 11. Hyperbolic metric and its properties.
- Week 12. Denjoy-Wolf theorem. Sullivan classification of Fatou components (proof).
- Week 13. Sullivan classification (continued).
- Week 14-15. Survey: baby Mandelbrots, renormalization at the Feigenbaum parameter, Mitziurevich points and dendrites, local connectivity conjeture and implications.